Matrix notation for linear systems

Example We can simplify the clerical load in reducing this system

$$\begin{aligned} -3x \phantom{{}-2y} + 2z &= -1 \\ x - 2y + 2z &= -5/3 \\ -x - 4y + 6z &= -13/3 \end{aligned}$$

by writing it as an *augmented matrix*.

$$\left(\begin{array}{ccc|c} -3 & 0 & 2 & -1 \\ 1 & -2 & 2 & -5/3 \\ -1 & -4 & 6 & -13/3 \end{array}\right) \xrightarrow[-(1/3)\rho_1+\rho_3]{(1/3)\rho_1+\rho_2} \left(\begin{array}{ccc|c} -3 & 0 & 2 & -1 \\ 0 & -2 & 8/3 & -2 \\ 0 & -4 & 16/3 & -4 \end{array}\right)$$

$$\xrightarrow{-2\rho_2+\rho_3} \left(\begin{array}{ccc|c} -3 & 0 & 2 & -1 \\ 0 & -2 & 8/3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The two nonzero rows give $-3x + 2z = -1$ and $-2y + (8/3)z = -2$.

Parametrizing $-3x + 2z = -1$ and $-2y + (8/3)z = -2$ gives this.

$$\begin{aligned} x &= (1/3) + (2/3)z \\ y &= 1 + (4/3)z \\ z &= z \end{aligned}$$

We can write the solution set in vector notation.

$$\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2/3 \\ 4/3 \\ 1 \end{pmatrix} z \mid z \in \mathbb{R} \}$$

Example Reducing this system

$$\begin{aligned} x + 2y - z \phantom{{}+w} &= 2 \\ 2x - y - 2z + w &= 5 \end{aligned}$$

using the augmented matrix notation

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 2 & -1 & -2 & 1 & 5 \end{array}\right) \xrightarrow{-2\rho_1+\rho_2} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 0 & -5 & 0 & 1 & 1 \end{array}\right)$$

gives this vector description of the solution set.

$$\{ \begin{pmatrix} 12/5 \\ -1/5 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} z + \begin{pmatrix} -2/5 \\ 1/5 \\ 0 \\ 1 \end{pmatrix} w \mid z, w \in \mathbb{R} \}$$

$$\text{General} = \text{Particular} + \text{Homogeneous}$$

Form of solution sets

Example This system

$$\begin{aligned} x + 2y - z \phantom{{}+w} &= 2 \\ 2x - y - 2z + w &= 5 \end{aligned}$$

has solutions of this form.

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 12/5 \\ -1/5 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} z + \begin{pmatrix} -2/5 \\ 1/5 \\ 0 \\ 1 \end{pmatrix} w \qquad z, w \in \mathbb{R}$$

Taking $z = w = 0$ shows that the first vector is a particular solution of the system.

3.2 *Definition* A linear equation is *homogeneous* if it has a constant of zero, so that it can be written as $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$.

Example From the above system we get this associated system of homogeneous equations by changing the constants to 0's.

$$\begin{aligned} x + 2y - \ \ z \qquad &= 0 \\ 2x - \ \ y - 2z + w &= 0 \end{aligned}$$

The same Gauss's Method steps reduce it to echelon form.

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 0 \\ 2 & -1 & -2 & 1 & 0 \end{array}\right) \xrightarrow{-2\rho_1+\rho_2} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 0 \\ 0 & -5 & 0 & 1 & 0 \end{array}\right)$$

The vector description of the solution set

$$\{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} z + \begin{pmatrix} -2/5 \\ 1/5 \\ 0 \\ 1 \end{pmatrix} w \mid z, w \in \mathbb{R}\}$$

is the same as earlier but with a particular solution that is the zero vector.