Example

$$\begin{array}{rcl} x - y + 2z + 3w = 14 \\ 2x - 2y - z + 2w = 6 \\ -3z + 2w = 0 \end{array} \xrightarrow{-2\rho_1+\rho_2} \begin{array}{rcl} x - y + 2z + 3w = 14 \\ -5z - 4w = -22 \\ -3z + 2w = 0 \end{array}$$

$$\xrightarrow{-(3/5)\rho_2+\rho_3} \begin{array}{rcl} x - y + 2z + 3w = 14 \\ -5z - 4w = -22 \\ (22/5)w = 66/5 \end{array}$$

The leading variables are x, z, and w. We will parametrize with the free variable y.

The bottom row gives $w = 3$ and substituting that into the next row up gives $z = 2$. The top equation is $x - y + 2 \cdot 2 + 3 \cdot 3 = 14$ so we have $x = 1 - y$.

$$\begin{aligned} x &= 1 - y \\ y &= y \\ z &= 2 \\ w &= 3 \end{aligned}$$

Matrices and vectors

2.6 *Definition* An $m \times n$ *matrix* is a rectangular array of numbers with m *rows* and n *columns*. Each number in the matrix is an *entry*.

Example This is a 2×3 matrix

$$B = \begin{pmatrix} 1 & -2 & 3 \\ 4 & -5 & 6 \end{pmatrix}$$

because it has 2 rows and 3 columns. The entry in row 2 and column 1 is $b_{2,1} = 4$.

2.8 *Definition* A *column vector*, often just called a *vector*, is a matrix with a single column. A matrix with a single row is a *row vector*. The entries of a vector are its *components*. A column or row vector whose components are all zeros is a *zero vector*.

We denote a vector with an over-arrow (many authors use boldface).

Example This column vector has three components.

$$\vec{v} = \begin{pmatrix} -1 \\ -0.5 \\ 0 \end{pmatrix}$$

Example This row vector has three components

$$\vec{w} = \begin{pmatrix} -1 & -0.5 & 0 \end{pmatrix}$$

Example This is the two-component zero vector.

$$\vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Vector operations

2.10 *Definition* The *vector sum* of $\vec{u}$ and $\vec{v}$ is the vector of the sums.

$$\vec{u}+\vec{v}=\begin{pmatrix}u_1\\ \vdots\\ u_n\end{pmatrix}+\begin{pmatrix}v_1\\ \vdots\\ v_n\end{pmatrix}=\begin{pmatrix}u_1+v_1\\ \vdots\\ u_n+v_n\end{pmatrix}$$

2.11 *Definition* The *scalar multiplication* of the real number r and the vector $\vec{v}$ is the vector of the multiples.

$$r\cdot\vec{v}=r\cdot\begin{pmatrix}v_1\\ \vdots\\ v_n\end{pmatrix}=\begin{pmatrix}rv_1\\ \vdots\\ rv_n\end{pmatrix}$$

Example

$$3\begin{pmatrix}1\\ 2\end{pmatrix}-2\begin{pmatrix}0\\ 1\end{pmatrix}=\begin{pmatrix}3\\ 4\end{pmatrix}$$

Matrix notation for linear systems

Example We can simplify the clerical load in reducing this system

$$\begin{aligned} -3x \phantom{{}-2y} + 2z &= -1 \\ x - 2y + 2z &= -5/3 \\ -x - 4y + 6z &= -13/3 \end{aligned}$$

by writing it as an *augmented matrix*.

$$\left(\begin{array}{ccc|c} -3 & 0 & 2 & -1 \\ 1 & -2 & 2 & -5/3 \\ -1 & -4 & 6 & -13/3 \end{array}\right) \xrightarrow[-(1/3)\rho_1+\rho_3]{(1/3)\rho_1+\rho_2} \left(\begin{array}{ccc|c} -3 & 0 & 2 & -1 \\ 0 & -2 & 8/3 & -2 \\ 0 & -4 & 16/3 & -4 \end{array}\right)$$

$$\xrightarrow{-2\rho_2+\rho_3} \left(\begin{array}{ccc|c} -3 & 0 & 2 & -1 \\ 0 & -2 & 8/3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The two nonzero rows give $-3x + 2z = -1$ and $-2y + (8/3)z = -2$.