Example Consider the set of row vectors consisting of all multiples of $(1 \quad 2)$.

$$V = \{(a \quad 2a) \mid a \in \mathbb{R}\}$$

Some members of V are $(4 \quad 8)$, $(1/2 \quad 1)$, $(-100 \quad -200)$, and $(0 \quad 0)$.

This V is a vector space under the natural addition

$$(a_1 \quad 2a_1) + (a_2 \quad 2a_2) = (a_1 + a_2 \quad 2a_1 + 2a_2)$$

and scalar multiplication operations.

$$r(a_1 \quad 2a_1) = (ra_1 \quad 2ra_1)$$

To verify that, we will check each of the ten conditions. Because this is the first time through the definition, we will verify these at length.

We first check closure under addition (1), that the sum of two members of V is also a member of V. Take $\vec{v}$ and $\vec{w}$ to be members of V.

$$\vec{v} = (v_1 \;\; 2v_1) \qquad \vec{w} = (w_1 \;\; 2w_1)$$

Then their sum

$$\vec{v} + \vec{w} = (v_1 + w_1 \;\; 2v_1 + 2w_1)$$

is also a member of V because its second entry is twice its first.

Condition (2), commutativity of addition, is straightforward. The sums in the two orders are

$$\vec{v} + \vec{w} = (v_1 + w_1 \;\; 2(v_1 + w_1))$$

and

$$\vec{w} + \vec{v} = (w_1 + v_1 \;\; 2(w_1 + v_1))$$

and the two are equal because $v_1 + w_1$ equals $w_1 + v_1$, as both are sums of real numbers and real number addition is commutative.

Condition (3), associativity of addition, is like the prior one. The left side is

$$(\vec{v}+\vec{w})+\vec{u}=((v_1+w_1)+u_1 \quad (2v_1+2w_1)+2u_1)$$

while the right side is this.

$$\vec{v}+(\vec{w}+\vec{u})=(v_1+(w_1+u_1) \quad 2v_1+(2w_1+2u_1))$$

The two are equal because real number addition is associative $(v_1+w_1)+u_1=v_1+(w_1+u_1)$.

For condition (4) we can just exhibit the member of V with the desired property. So consider $\vec{0}=(0 \quad 0)$. It is a member of V since its second component is twice its first. Note that it is the required identity element with respect to addition.

$$\begin{aligned}\vec{v}+\vec{0}&=(v_1 \quad 2v_1)+(0 \quad 0)\\&=(v_1 \quad 2v_1)\\&=\vec{v}\end{aligned}$$

Condition (5), existence of an additive inverse, is also a matter of producing the desired element. Given a member $\vec{v} = (v_1 \;\; 2v_1)$ of V, consider $\vec{w} = (-v_1 \;\; -2v_1)$. Then $\vec{w} \in V$, and note that it cancels $\vec{v}$.

$$\vec{w} + \vec{v} = (-v_1 \;\; -2v_1) + (v_1 \;\; 2v_1) = \vec{0}$$

We finish by verifying the five conditions having to do with scalar multiplication.

Condition (6) is closure under scalar multiplication. Consider a scalar $r \in \mathbb{R}$ and a vector $\vec{v} = (v_1 \;\; 2v_1) \in V$. The scalar multiple $r\vec{v} = (rv_1 \;\; r2v_1)$ is also a member of V because the second component is twice the first.

Condition (7) is that real number addition distributes over scalar multiplication. Let the scalars be $r, s \in \mathbb{R}$, and let the vector be $\vec{v} = (v_1 \;\; 2v_1) \in V$. Here is the check.

$$\begin{aligned}(r+s)\vec{v} &= ((r+s)v_1 \;\; (r+s)2v_1) \\ &= (rv_1 \;\; 2rv_1) + (sv_1 \;\; 2sv_1) \\ &= r\vec{v} + s\vec{v}\end{aligned}$$

For (8), distributivity of vector addition over scalar multiplication, take a scalar $r \in \mathbb{R}$ and two vectors $\vec{v}, \vec{w} \in V$.

$$\begin{aligned} r(\vec{v} + \vec{w}) &= (rv_1 \;\; 2rv_1) + (rw_1 \;\; 2rw_1) \\ &= (rv_1 + rw_1 \;\; 2rv_1 + 2rw_1) \\ &= r(v_1 \;\; 2v_1) + r(w_1 \;\; 2w_1) \\ &= r\vec{v} + r\vec{w} \end{aligned}$$

For condition (9) suppose $r, s \in \mathbb{R}$ and $\vec{v} = (v_1 \;\; 2v_1) \in V$. The left side is $(rs)(v_1 \;\; 2v_1) = ((rs)v_1 \;\; (rs)2v_1)$, while the right side is $r(s(v_1 \;\; 2v_1)) = r(sv_1 \;\; s2v_1) = (r(sv_1) \;\; r(s2v_1))$. The two are equal because $(rs)v_1 = r(sv_1)$ and $(rs)2v_1 = r(s2v_1)$, as those are real number multiplications.

Condition (10) is simple: $1\vec{v} = 1(v_1 \;\; 2v_1) = (1 \cdot v_1 \;\; 1 \cdot 2v_1) = \vec{v}$ for any $\vec{v} \in V$.

Therefore the set $V = \{(a \;\; 2a) \mid a \in \mathbb{R}\}$ is a vector space under the natural addition and scalar multiplication operations.

Example This plane through the origin subset of $\mathbb{R}^3$

$$P = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid 2x + y + 3z = 0 \}$$

is a vector space. We will verify conditions (1) and (6) (the others are exactly as in the prior example).

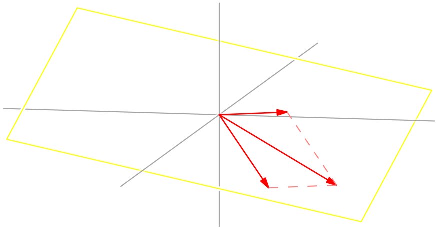
For (1) suppose that these are members of the plane

$$\vec{p}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \quad \vec{p}_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

so that both $2x_1 + y_1 + 3z_1 = 0$ and $2x_2 + y_2 + 3z_2 = 0$. Then the sum is

$$\vec{p}_1 + \vec{p}_2 = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}$$

and to verify that it is in the plane note that $2(x_1+x_2)+(y_1+y_2)+3(z_1+z_2) = (2x_1+y_1+3z_1)+(2x_2+y_2+3z_2) = 0$.



For condition (6) take a member of the plane

$$\vec{p} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \qquad \text{such that } 2x_1 + y_1 + 3z_1 = 0$$

and multiply by a scalar $r \in \mathbb{R}$.

$$r\vec{p} = \begin{pmatrix} rx_1 \\ ry_1 \\ rz_1 \end{pmatrix}$$

Verify that $r\vec{p}$ is a member of the plane P with $2(rx_1) + (ry_1) + 3(rz_1) = r(2x_1 + y_1 + 3z_1) = 0$.

Example The set $\mathcal{P}_2 = \{a_0 + a_1x + a_2x^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$ of quadratic polynomials is a vector space under the usual operations of polynomial addition

$$(a_0+a_1x+a_2x^2)+(b_0+b_1x+b_2x^2) = (a_0+b_0)+(a_1+b_1)x+(a_2+b_2)x^2$$

and scalar multiplication.

$$r \cdot (a_0 + a_1x + a_2x^2) = (ra_0) + (ra_1)x + (ra_2)x^2$$

We won't here check all the conditions but in particular note that this space is closed: a linear combination of quadratic polynomials is a quadratic polynomial. For instance, here is a sample combination in $\mathcal{P}_2$:

$$4 \cdot (1 + 2x + 3x^2) - (1/5) \cdot (10 + 5x^2) = 2 + 8x + 11x^2$$

a linear combination of quadratic polynomials is a quadratic polynomial.

Example The set of 3×3 matrices

$$\mathcal{M}_{3\times 3} = \{ \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \mid a_{i,j} \in \mathbb{R} \}$$

is a vector space under the usual matrix addition and scalar multiplication. The check of the ten conditions is straightforward.

Here is a sample linear combination.

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ -1 & 3 & 1/2 \end{pmatrix} - 3 \begin{pmatrix} 0 & 0 & 2 \\ 1 & 1 & 1 \\ 0 & 4 & 3/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -5 \\ -1 & -3 & 1 \\ -1 & -9 & -4 \end{pmatrix}$$

The empty set cannot be made a vector space, regardless of which operations we use, because the definition requires that the space contains an additive identity.

Example The set consisting only of the two-tall zero vector

$$V = \{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \}$$

is a vector space (under the usual vector addition and scalar multiplication operations).

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad r \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

1.7 *Definition* A one-element vector space is a *trivial* space.

1.16 *Lemma* In any vector space V, for any $\vec{v} \in V$ and $r \in \mathbb{R}$, we have (1) $0 \cdot \vec{v} = \vec{0}$, (2) $(-1 \cdot \vec{v}) + \vec{v} = \vec{0}$, and (3) $r \cdot \vec{0} = \vec{0}$.

Proof For (1) note that $\vec{v} = (1+0) \cdot \vec{v} = \vec{v} + (0 \cdot \vec{v})$. Add to both sides the additive inverse of $\vec{v}$, the vector $\vec{w}$ such that $\vec{w} + \vec{v} = \vec{0}$.

$$\begin{aligned} \vec{w} + \vec{v} &= \vec{w} + \vec{v} + 0 \cdot \vec{v} \\ \vec{0} &= \vec{0} + 0 \cdot \vec{v} \\ \vec{0} &= 0 \cdot \vec{v} \end{aligned}$$

Item (2) is easy: $(-1 \cdot \vec{v}) + \vec{v} = (-1+1) \cdot \vec{v} = 0 \cdot \vec{v} = \vec{0}$. For (3), $r \cdot \vec{0} = r \cdot (0 \cdot \vec{0}) = (r \cdot 0) \cdot \vec{0} = \vec{0}$ will do. QED

Subspaces and spanning sets

Subspace

2.1 *Definition* For any vector space, a *subspace* is a subset that is itself a vector space, under the inherited operations.

Any vector space has a trivial subspace $\{\vec{0}\,\}$. At the opposite extreme, any vector space has itself for a subspace. A subspace that is not the entire space is a *proper* subspace.

Example In the vector space $\mathbb{R}^2$, the line $y = 2x$

$$S = \{\begin{pmatrix} a \\ 2a \end{pmatrix} \mid a \in \mathbb{R}\} = \{\begin{pmatrix} 1 \\ 2 \end{pmatrix} a \mid a \in \mathbb{R}\}$$

is a subspace. The operations, as required by the definition above, are the ones from $\mathbb{R}^2$. We could show it is a vector space by checking the ten conditions but the next result gives an easier way.

Example This subset of $\mathcal{M}_{2\times 2}$ is a subspace.

$$S = \{\begin{pmatrix} a & b \\ a & b \end{pmatrix} \mid a, b \in \mathbb{R}\} = \{\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} a + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} b \mid a, b \in \mathbb{R}\}$$

As above, addition and scalar multiplication are the same as in $\mathcal{M}_{2\times 2}$.

2.9 *Lemma* For a nonempty subset S of a vector space, under the inherited operations the following are equivalent statements.

(1) S is a subspace of that vector space

(2) S is closed under linear combinations of pairs of vectors: for any vectors $\vec{s}_1, \vec{s}_2 \in S$ and scalars r_1, r_2 the vector $r_1\vec{s}_1 + r_2\vec{s}_2$ is in S

(3) S is closed under linear combinations of any number of vectors: for any vectors $\vec{s}_1, \ldots, \vec{s}_n \in S$ and scalars $r_1, \ldots, r_n$ the vector $r_1\vec{s}_1 + \cdots + r_n\vec{s}_n$ is an element of S.

The book has the full proof. Its idea is that if V is a vector space with a subset S then many of the ten properties required for S to be a vector space are automatic. For instance, suppose that $\vec{s}_1, \vec{s}_2 \in S$ and consider commutativity of addition: does $\vec{s}_1 + \vec{s}_2$ equal $\vec{s}_2 + \vec{s}_1$? Because the $+$ operation is inherited from V and as sums of elements of V the two are equal $\vec{s}_1 + \vec{s}_2 = \vec{s}_2 + \vec{s}_1$, then provided S is closed the two are equal in S.

Many of the other nine conditions are also automatic. The only ones that need to be checked are the closure conditions. Both statements (2) and (3) above just combine the two closure conditions into a single one, to make the subspace verification faster.