Two.I Vector Space Definition

Linear Algebra

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Definition and examples

Vector space

1.1 *Definition* A *vector space* (over $\mathbb{R}$) consists of a set V along with two operations '+' and '$\cdot$' subject to the conditions that for all vectors $\vec{v}, \vec{w}, \vec{u} \in V$, and all *scalars* $r, s \in \mathbb{R}$:

1) the set V is *closed* under vector addition, that is, $\vec{v} + \vec{w} \in V$
2) vector addition is commutative $\vec{v} + \vec{w} = \vec{w} + \vec{v}$
3) vector addition is associative $(\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u})$
4) there is a *zero vector* $\vec{0} \in V$ such that $\vec{v} + \vec{0} = \vec{v}$ for all $\vec{v} \in V$
5) each $\vec{v} \in V$ has an *additive inverse* $\vec{w} \in V$ such that $\vec{w} + \vec{v} = \vec{0}$
6) the set V is closed under scalar multiplication, that is, $r \cdot \vec{v} \in V$
7) addition of scalars distributes over scalar multiplication $(r + s) \cdot \vec{v} = r \cdot \vec{v} + s \cdot \vec{v}$
8) scalar multiplication distributes over vector addition $r \cdot (\vec{v} + \vec{w}) = r \cdot \vec{v} + r \cdot \vec{w}$
9) ordinary multipication of scalars associates with scalar multiplication $(rs) \cdot \vec{v} = r \cdot (s \cdot \vec{v})$
10) multiplication by the scalar 1 is the identity operation $1 \cdot \vec{v} = \vec{v}$.

Example The set $\mathbb{R}^3$ is a vector space under the usual vector addition and scalar multiplication operations.

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{pmatrix} \quad \text{and} \quad r \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} rv_1 \\ rv_2 \\ rv_3 \end{pmatrix}$$

To verify that, we will check the conditions (more briefly than for the prior example).

Condition (1) is closure under addition. This is clear because the only condition for membership in the set $\mathbb{R}^3$ is to be a three-tall vector of reals, and the sum of two three-tall vectors of reals is also a three-tall vector of reals.

Condition (2) is routine.

$$\vec{v} + \vec{w} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \vec{w} + \vec{v}$$

Condition (3) is also a direct consequence of the related real number property.

$$(\vec{v}+\vec{w})+\vec{u}=\begin{pmatrix} v_1+w_1 \\ v_2+w_2 \\ v_3+w_3 \end{pmatrix}+\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}=\begin{pmatrix} v_1+w_1+u_1 \\ v_2+w_2+u_2 \\ v_3+w_3+u_3 \end{pmatrix}$$
$$=\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}+\begin{pmatrix} w_1+u_1 \\ w_2+u_2 \\ w_3+u_3 \end{pmatrix}=\vec{v}+(\vec{w}+\vec{u})$$

For condition (4) take the vector of 0's.

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}+\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}=\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

For condition (5), given $\vec{v}\in\mathbb{R}^3$, use $\vec{w}=-1\vec{v}$ as the additive inverse.

$$\begin{pmatrix} -v_1 \\ -v_2 \\ -v_3 \end{pmatrix}+\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Condition (6) is closure under scalar multiplication. Let the scalar be $r \in \mathbb{R}$ and the vector be $\vec{v} \in \mathbb{R}^3$. Then $r\vec{v}$ is a three-tall vector of reals, so $r\vec{v} \in \mathbb{R}^3$.

Conditions (7)

$$(r+s)\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} (r+s)v_1 \\ (r+s)v_2 \\ (r+s)v_3 \end{pmatrix} = \begin{pmatrix} rv_1 + sv_1 \\ rv_2 + sv_2 \\ rv_3 + sv_3 \end{pmatrix} = \begin{pmatrix} rv_1 \\ rv_2 \\ rv_3 \end{pmatrix} + \begin{pmatrix} sv_1 \\ sv_2 \\ sv_3 \end{pmatrix} = r\vec{v}+s\vec{v}$$

and (8)

$$r(\vec{v}+\vec{w}) = r(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}) = r\begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{pmatrix} = \begin{pmatrix} rv_1 + rw_1 \\ rv_2 + rw_2 \\ rv_3 + rw_3 \end{pmatrix} = r\vec{v}+r\vec{w}$$

are straightforward.

Condition (9) is similar.

$$(rs)\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} (rs)v_1 \\ (rs)v_2 \\ (rs)v_3 \end{pmatrix} = r\begin{pmatrix} sv_1 \\ sv_2 \\ sv_3 \end{pmatrix} = r(s\vec{v})$$

And (10) is also easy.

$$1\vec{v} = 1\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 \cdot v_1 \\ 1 \cdot v_2 \\ 1 \cdot v_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \vec{v}$$

So the set $\mathbb{R}^3$ is a vector space under the usual operations of vector addition and scalar-vector multiplication.