Dot product

2.3 *Definition* The *dot product* (or *inner product* or *scalar product*) of two n-component real vectors is the linear combination of their components.

$$\vec{u} \bullet \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

Example The dot product of two vectors

$$\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -3 \\ 4 \end{pmatrix} = 3 - 3 - 4 = -4$$

is a scalar, not a vector.

The dot product of a vector with itself $\vec{v} \bullet \vec{v} = v_1^2 + \cdots + v_n^2$ is the square of the vector's length.

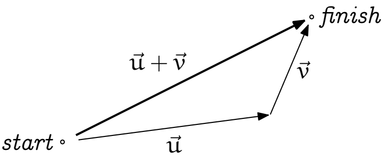
Triangle Inequality

2.5 *Theorem* For any $\vec{u}, \vec{v} \in \mathbb{R}^n$,

$$|\vec{u} + \vec{v}| \leqslant |\vec{u}| + |\vec{v}|$$

with equality if and only if one of the vectors is a nonnegative scalar multiple of the other one.

This is the source of the familiar saying, "The shortest distance between two points is in a straight line."

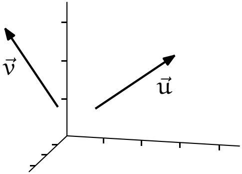


Angle measure

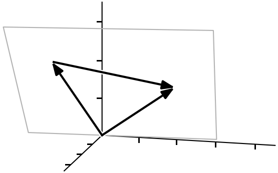
Definition The *angle* between two vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ is this.

$$\theta = \arccos\left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}\,|\,|\vec{v}\,|} \right)$$

We motivate that definition with two vectors in $\mathbb{R}^3$.



If neither is a multiple of the other then they determine a plane, because if we put them in canonical position then the origin and the endpoints make three noncolinear points. Consider the triangle formed by $\vec{u}$, $\vec{v}$, and $\vec{u} - \vec{v}$.



Apply the Law of Cosines: $|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2\,|\vec{u}|\,|\vec{v}|\cos\theta$ where θ is the angle that we want to find. The left side gives

$$\begin{aligned}&(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2 \\ &\qquad = (u_1^2 - 2u_1v_1 + v_1^2) + (u_2^2 - 2u_2v_2 + v_2^2) + (u_3^2 - 2u_3v_3 + v_3^2)\end{aligned}$$

while the right side gives this.

$$(u_1^2 + u_2^2 + u_3^2) + (v_1^2 + v_2^2 + v_3^2) - 2\,|\vec{u}|\,|\vec{v}|\cos\theta$$

Canceling squares $u_1^2\ \ldots,\ v_3^2$ and dividing by 2 gives the formula.

2.8 *Corollary* Vectors from $\mathbb{R}^n$ are orthogonal, that is, perpendicular, if and only if their dot product is zero. They are parallel if and only if their dot product equals the product of their lengths.