One.II Linear Geometry

Linear Algebra

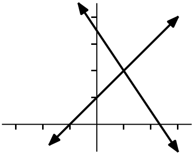
Jim Hefferon

http://joshua.smcvt.edu/linearalgebra

Geometry

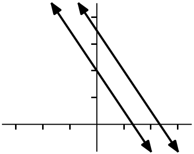
We can draw two-unknown equations as lines. Then the three possibilities for solution sets become clear.

Unique solution



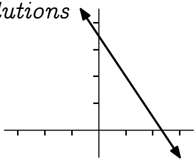
$$\begin{aligned} 3x + 2y &= 7 \\ x - y &= -1 \end{aligned}$$

No solutions



$$\begin{aligned} 3x + 2y &= 7 \\ 3x + 2y &= 4 \end{aligned}$$

Infinitely many solutions



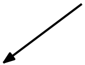
$$\begin{aligned} 3x + 2y &= 7 \\ 6x + 4y &= 14 \end{aligned}$$

Besides being pretty, the geometry helps us understand what is happening.

Vectors in space

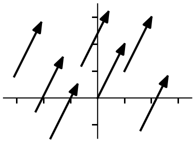
Vectors

A *vector* is an object consisting of a magnitude and a direction.



For instance, a vector can model a displacement.

Two vectors with the same magnitude and same direction, such as all of these, are equal.



For instance, each of the above could model a displacement of one over and two up.

Denote the vector that extends from (a_1, a_2) to (b_1, b_2) by

$$\begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}$$

so the "one over, two up" vector would be written in this way.

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

We often picture a vector

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

as starting at the origin. From there $\vec{v}$ extends to (v_1, v_2) and we may refer to it as "the point $\vec{v}$" so that we may call each of these $\mathbb{R}^2$.

$$\{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\} \qquad \{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mid x_1, x_2 \in \mathbb{R}\}$$

These definitions extend to higher dimensions. The vector that starts at $(a_1, \ldots, a_n)$ and ends at $(b_1, \ldots, b_n)$ is represented by this column

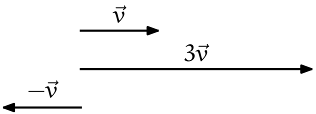
$$\begin{pmatrix} b_1 - a_1 \\ \vdots \\ b_n - a_n \end{pmatrix}$$

and two vectors are equal if they have the same representation. Also, we aren't too careful about distinguishing between a point and the vector which, when it starts at the origin, ends at that point.

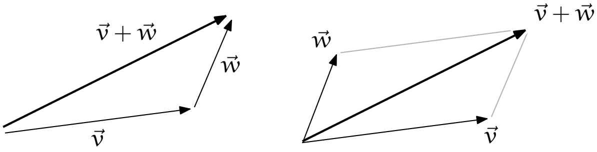
$$\mathbb{R}^n = \{ \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \mid v_1, \ldots, v_n \in \mathbb{R} \}$$

Vector operations

Scalar multiplication makes a vector longer or shorter, including possibly flipping it around.



Where $\vec{v}$ and $\vec{w}$ represent displacements, the vector sum $\vec{v} + \vec{w}$ represents those displacements combined.

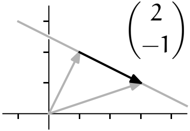


The second drawing shows the *parallelogram rule* for vector addition.

Lines

The line in $\mathbb{R}^2$ through $(1,2)$ and $(3,1)$ is comprised of the vectors in this set

$$\{\begin{pmatrix}1\\2\end{pmatrix}+t\begin{pmatrix}2\\-1\end{pmatrix} \mid t\in\mathbb{R}\}$$



$$\begin{pmatrix}2\\-1\end{pmatrix}=\begin{pmatrix}3\\1\end{pmatrix}-\begin{pmatrix}1\\2\end{pmatrix}$$

(that is, it is comprised of the endpoints of those vectors). The vector associated with the parameter t

$$\begin{pmatrix}2\\-1\end{pmatrix}$$

is a *direction vector* for the line. Lines in higher dimensions work the same way.