3.6 *Lemma* For any homogeneous linear system there exist vectors $\vec{\beta}_1$, ..., $\vec{\beta}_k$ such that the solution set of the system is

$$\{c_1\vec{\beta}_1 + \cdots + c_k\vec{\beta}_k \mid c_1, \ldots, c_k \in \mathbb{R}\}$$

where $k$ is the number of free variables in an echelon form version of the system.

*Example* The book has the full proof. For the central idea consider this system of homogeneous equations.

$$\begin{aligned} x + y + z + w &= 0 \\ y - z + w &= 0 \end{aligned}$$

Using the bottom equation, express the leading variable $y$ in terms of the free variables $y = z - w$. Next, move up, substitute $x + (z - w) + z + w = 0$, and solve for the leading variable $x = -2z$. Finish by describing the solution in vector notation.

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} z + \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} w \qquad z, w \in \mathbb{R}$$

and recognize the vectors associated with $z$ and $w$ as $\vec{\beta}_1$ and $\vec{\beta}_2$.

3.7 *Lemma* For a linear system and for any particular solution $\vec{p}$, the solution set equals $\{\vec{p}+\vec{h} \mid \vec{h} \text{ satisfies the associated homogeneous system}\}$.

3.7 *Proof* For set inclusion the first way, that if a vector solves the system then it is in the set described above, assume that $\vec{s}$ solves the system. Then $\vec{s}-\vec{p}$ solves the associated homogeneous system since for each equation index $i$,

$$\begin{aligned}&a_{i,1}(s_1-p_1)+\cdots+a_{i,n}(s_n-p_n)\\&=(a_{i,1}s_1+\cdots+a_{i,n}s_n)-(a_{i,1}p_1+\cdots+a_{i,n}p_n)=d_i-d_i=0\end{aligned}$$

where $p_j$ and $s_j$ are the $j$-th components of $\vec{p}$ and $\vec{s}$. Express $\vec{s}$ in the required $\vec{p}+\vec{h}$ form by writing $\vec{s}-\vec{p}$ as $\vec{h}$.

For set inclusion the other way, take a vector of the form $\vec{p}+\vec{h}$, where $\vec{p}$ solves the system and $\vec{h}$ solves the associated homogeneous system and note that $\vec{p}+\vec{h}$ solves the given system since for any equation index $i$,

$$\begin{aligned} &a_{i,1}(p_1+h_1)+\cdots+a_{i,n}(p_n+h_n)\\ &=(a_{i,1}p_1+\cdots+a_{i,n}p_n)+(a_{i,1}h_1+\cdots+a_{i,n}h_n)=d_i+0=d_i \end{aligned}$$

where as earlier $p_j$ and $h_j$ are the $j$-th components of $\vec{p}$ and $\vec{h}$. QED

3.1 *Theorem* Any linear system's solution set has the form

$$\{\vec{p} + c_1\vec{\beta}_1 + \cdots + c_k\vec{\beta}_k \mid c_1, \ldots, c_k \in \mathbb{R}\}$$

where $\vec{p}$ is any particular solution and where the number of vectors $\vec{\beta}_1, \ldots, \vec{\beta}_k$ equals the number of free variables that the system has after a Gaussian reduction.

*Proof* This restates the prior two lemmas. QED

3.10 *Corollary* Solution sets of linear systems are either empty, have one element, or have infinitely many elements.

The book contains the full proof.