Name:

Problem 1: *Find the dimension of the vector space of matrices*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

such that

$$\begin{aligned} a+b+c=0 \\ a+b-c=0 \end{aligned}$$

and $d \in \mathbb{R}$.
Show some work to justify your answer.
Tip: You can pretend the matrices are just vectors in $\mathbb{R}^4$ *if you need.*

Solution: To find the dimension, we find a basis. To find a basis, we must first find a spanning set, and then we will show that the spanning set is linearly independent.
To find a spanning set, we solve the two equations given, and write the answer in vector form:

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{pmatrix} \overset{\rho_2-\rho_1}{\sim} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{pmatrix} \overset{-\frac{1}{2}\rho_2}{\sim} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \overset{\rho_1-\rho_2}{\sim} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We see that b and d are free and a and c are leading variables. The solutions are

$$a = -b, \quad c = 0, \quad \text{and} b, d \in \mathbb{R}.$$

In vector form this is:

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} b + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} d.$$

Since we are dealing with matrices, what we really want to say is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} b + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} d.$$

Therefore the space we are speaking of is

$$\text{Span}\left\{ \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

These two matrices are linearly independent since if

$$a_1 \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

then $a_1 = a_2 = 0$ to make the top right and the bottom right entries 0.
Therefore this space has dimension 2.