Name:

Problem 1: *Find a basis for, and the dimension of, the space of solutions to the system*

$$\begin{aligned} x_1 - 4x_2 + 3x_3 - \ \ x_4 &= 0 \\ 2x_1 - 8x_2 + 6x_3 - 2x_4 &= 0 \end{aligned}$$

Do not forget to prove that you really have found a basis.

Solution: As per the hint, we start by solving the system of equations:

$$\begin{pmatrix} 1 & -4 & 3 & -1 \\ 2 & -8 & 6 & -2 \end{pmatrix} \overset{\rho_2 - 2\rho_1}{\sim} \begin{pmatrix} 1 & -4 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This is in reduced echelon form. We see that x_1 is a leading variable and the other three variables are free. This means that the solution set has dimension 3.

To get a basis, we write the answer in vector form. The vectors we get will span the solution space, and then we will check that they are linearly independent and therefore also a basis.

The solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} x_4.$$

The solution set is thus

$$\text{Span}\left\{ \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

We now see if we can throw out any vectors (we shouldn't because we have three vectors and the solution space has dimension 3, but we still have to check to be sure that we didn't make a mistake). We try and solve

$$\begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix} a_1 + \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} a_2 + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} a_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Looking at the last three entries of the vector, we see that this forces $a_1 = a_2 = a_3 = 0$ so the vectors really are linearly independent, and therefore a basis.