Two.II Linear Independence

Linear Algebra

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Definition and examples

Linear independence

1.4 *Definition* A multiset subset of a vector space is *linearly independent* if none of its elements is a linear combination of the others. Otherwise it is *linearly dependent*.

Observe that, although this way of writing one vector as a combination of the others

$$\vec{s}_0 = c_1\vec{s}_1 + c_2\vec{s}_2 + \cdots + c_n\vec{s}_n$$

visually sets off $\vec{s}_0$, algebraically there is nothing special about that vector in that equation. For any $\vec{s}_i$ with a coefficient c_i that is non-0 we can rewrite to isolate $\vec{s}_i$.

$$\vec{s}_i = (1/c_i)\vec{s}_0 + \cdots + (-c_{i-1}/c_i)\vec{s}_{i-1} + (-c_{i+1}/c_i)\vec{s}_{i+1} + \cdots + (-c_n/c_i)\vec{s}_n$$

When we don't want to single out any vector we will instead say that $\vec{s}_0, \vec{s}_1, \ldots, \vec{s}_n$ are in a *linear relationship* and put all of the vectors on the same side.

1.5 *Lemma* A subset S of a vector space is linearly independent if and only if among its elements the only linear relationship $c_1\vec{s}_1 + \cdots + c_n\vec{s}_n = \vec{0}$ (with $\vec{s}_i \neq \vec{s}_j$ for all $i \neq j$) is the trivial one $c_1 = 0, \ldots, c_n = 0$.

Proof If S is linearly independent then no vector $\vec{s}_i$ is a linear combination of other vectors from S so there is no linear relationship where some of the $\vec{s}$'s have nonzero coefficients.

If S is not linearly independent then some $\vec{s}_i$ is a linear combination $\vec{s}_i = c_1\vec{s}_1 + \cdots + c_{i-1}\vec{s}_{i-1} + c_{i+1}\vec{s}_{i+1} + \cdots + c_n\vec{s}_n$ of other vectors from S. Subtracting $\vec{s}_i$ from both sides gives a relationship involving a nonzero coefficient, the -1 in front of $\vec{s}_i$. QED

So to decide if a list of vectors $\vec{s}_0, \ldots, \vec{s}_n$ is linearly independent, set up the equation $\vec{0} = c_0\vec{s}_0 + \cdots + c_n\vec{s}_n$, and calculate whether it has any solutions other than the trivial one where all coefficents are zero.

Example This set of vectors in the plane $\mathbb{R}^2$ is linearly independent.

$$\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\}$$

The only solution to this equation

$$c_1\begin{pmatrix}1\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

is trivial $c_1 = 0$, $c_2 = 0$.

Example In the vector space of cubic polynomials $\mathcal{P}_3 = \{a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_i \in \mathbb{R}\}$ the set $\{1 - x, 1 + x\}$ is linearly independent. Setting up the equation $c_0(1-x)+c_1(1+x) = 0$ and considering the constant term and linear term, leads to this system

$$\begin{aligned} c_0 + c_1 &= 0 \\ -c_0 + c_1 &= 0 \end{aligned}$$

which has only the trivial solution.

Example The nonzero rows of this matrix form a linearly independent set.

$$\begin{pmatrix} 2 & 0 & 1 & -1 \\ 0 & 1 & -3 & 1/2 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We showed in Lemma One.III.2.5 that in any echelon form matrix the nonzero rows form a linearly independent set.

Example This subset of $\mathbb{R}^3$ is linearly dependent.

$$\{ \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} \}$$

One way to see that is to spot that the third vector is twice the first plus the second. Another way is to solve the linear system

$$\begin{aligned} c_1 - c_2 + c_3 &= 0 \\ c_1 + c_2 + 3c_3 &= 0 \\ 3c_1 + 6c_3 &= 0 \end{aligned}$$

and note that it has more than just the trivial solution.