Vector Spaces and Linear Systems

Row space

3.1 *Definition* The *row space* of a matrix is the span of the set of its rows. The *row rank* is the dimension of this space, the number of linearly independent rows.

3.3 *Lemma* If two matrices A and B are related by a row operation

$$A \xrightarrow{\rho_i \leftrightarrow \rho_j} B \quad \text{or} \quad A \xrightarrow{k\rho_i} B \quad \text{or} \quad A \xrightarrow{k\rho_i + \rho_j} B$$

(for $i \neq j$ and $k \neq 0$) then their row spaces are equal. Hence, row-equivalent matrices have the same row space and therefore the same row rank.

Proof Corollary One.III.2.4 shows that when $A \longrightarrow B$ then each row of B is a linear combination of the rows of A. That is, in the above terminology, each row of B is an element of the row space of A. Then $\text{Rowspace}(B) \subseteq \text{Rowspace}(A)$ follows because a member of the set $\text{Rowspace}(B)$ is a linear combination of the rows of B, so it is a combination of combinations of the rows of A, and by the Linear Combination Lemma is also a member of $\text{Rowspace}(A)$.

Column space

3.6 *Definition* The *column space* of a matrix is the span of the set of its columns. The *column rank* is the dimension of the column space, the number of linearly independent columns.

Example This system

$$\begin{aligned} 2x + \quad 3y &= d_1 \\ -x + (1/2)y &= d_2 \end{aligned}$$

has a solution for those $d_1, d_2 \in \mathbb{R}$ that we can find to satisfy this vector equation.

$$x \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix} + y \cdot \begin{pmatrix} 3 \\ 1/2 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \qquad x, y \in \mathbb{R}$$

That is, the system has a solution if and only if the vector on the right is in the column space of this matrix.

$$\begin{pmatrix} 2 & 3 \\ -1 & 1/2 \end{pmatrix}$$

Example The column rank of this matrix

$$\begin{pmatrix} 2 & -1 & 3 & 1 & 0 & 1 \\ 3 & 0 & 1 & 1 & 4 & -1 \\ 4 & -2 & 6 & 2 & 0 & 2 \\ 1 & 0 & 3 & 0 & 0 & 2 \end{pmatrix}$$

is 3. Its largest set of linearly independent columns is size 3 because that's the size of its largest set of linearly independent rows.

$$\xrightarrow[\substack{-2\rho_1+\rho_3 \\ -(1/2)\rho_1+\rho_4}]{-(3/2)\rho_1+\rho_2} \xrightarrow{-(1/3)\rho_2+\rho_4} \xrightarrow{\rho_3\leftrightarrow\rho_4} \begin{pmatrix} 2 & -1 & 3 & 1 & 0 & 1 \\ 0 & 3/2 & -7/2 & -1/2 & 4 & -5/2 \\ 0 & 0 & 8/3 & -1/3 & -4/3 & 7/3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$