2.11 *Corollary* No linearly independent set can have a size greater than the dimension of the enclosing space.

Proof The proof of Theorem 2.5 never uses that D spans the space, only that it is linearly independent. QED

Remark This is an example of a result that assumes the vector spaces are finite-dimensional without specifically saying so.

2.13 *Corollary* Any linearly independent set can be expanded to make a basis.

Proof If a linearly independent set is not already a basis then it must not span the space. Adding to the set a vector that is not in the span will preserve linear independence by Lemma II.1.14 . Keep adding until the resulting set does span the space, which the prior corollary shows will happen after only a finite number of steps.

QED

2.14 *Corollary* Any spanning set can be shrunk to a basis.

Proof Call the spanning set S. If S is empty then it is already a basis (the space must be a trivial space). If $S = \{\vec{0}\}$ then it can be shrunk to the empty basis, thereby making it linearly independent, without changing its span.

Otherwise, S contains a vector $\vec{s}_1$ with $\vec{s}_1 \neq \vec{0}$ and we can form a basis $B_1 = \langle \vec{s}_1 \rangle$. If $[B_1] = [S]$ then we are done. If not then there is a $\vec{s}_2 \in [S]$ such that $\vec{s}_2 \notin [B_1]$. Let $B_2 = \langle \vec{s}_1, \vec{s}_2 \rangle$; by Lemma II.1.14 this is linearly independent so if $[B_2] = [S]$ then we are done.

We can repeat this process until the spans are equal, which must happen in at most finitely many steps. QED

2.15 *Corollary* In an n-dimensional space, a set composed of n vectors is linearly independent if and only if it spans the space.

Proof First we will show that a subset with n vectors is linearly independent if and only if it is a basis. The 'if' is trivially true — bases are linearly independent. 'Only if' holds because a linearly independent set can be expanded to a basis, but a basis has n elements, so this expansion is actually the set that we began with.

To finish, we will show that any subset with n vectors spans the space if and only if it is a basis. Again, 'if' is trivial. 'Only if' holds because any spanning set can be shrunk to a basis, but a basis has n elements and so this shrunken set is just the one we started with.

QED

Example This clearly spans the space.

$$\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rangle \subseteq \mathbb{R}^3$$

Because it has same number of elements as the dimension of the space, it is therefore a basis.