Dimension

Definition of dimension

2.1 *Definition* A vector space is *finite-dimensional* if it has a basis with only finitely many vectors.

Example The space $\mathbb{R}^3$ is finite-dimensional since it has a basis with three elements $\mathcal{E}_3$.

Example The space of quadratic polynomials $\mathcal{P}_2$ has at least one basis with finitely many elements, $\langle 1, x, x^2\rangle$, so it is finite-dimensional.

Example The space $\mathcal{M}_{2\times 2}$ of 2×2 matrices is finite-dimensional. Here is one basis with finitely many members.

$$\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \rangle$$

Note From this point on we will restrict our attention to vector spaces that are finite-dimensional. All the later examples, definitions, and theorems assume this of the spaces.

All of a space's bases are the same size

2.5 *Theorem* In any finite-dimensional vector space, all bases have the same number of elements.

Example The idea of the proof is that, given two bases, exchange members of the second for members of the first, which shows that they have the same number of elements. For an illustration, these are bases for $\mathcal{P}_2$.

$$B = \langle 1 + x + x^2, 1 + x, 1 \rangle \qquad D = \langle 2, 2x, 2x^2 \rangle$$

For the first element of D, represent it with respect to B, look for a nonzero coefficient, and exchange.

$$2 = 0 \cdot (1 + x + x^2) + 0 \cdot (1 + x) + 2 \cdot (1) \quad B_1 = \langle 1 + x + x^2, 1 + x, 2 \rangle$$

Iterate (only exchanging for elements of B).

$$2x = 0 \cdot (1 + x + x^2) + 2 \cdot (1 + x) - 1 \cdot (2) \quad B_2 = \langle 1 + x + x^2, 2x, 2 \rangle$$

The third exchange finishes it off.

$$2x^2 = 2 \cdot (1 + x + x^2) - 1 \cdot (2x) - 1 \cdot (2) \quad B_3 = D$$

Definition of dimension

2.6 *Definition* The *dimension* of a vector space is the number of vectors in any of its bases.

Example The vector space $\mathbb{R}^n$ has dimension n because that is how many members are in $\mathcal{E}_n$.

Example The vector space $\mathcal{P}_2$ has dimension 3 because one of its bases is $\langle 1, x, x^2 \rangle$. More generally, $\mathcal{P}_n$ has dimension $n+1$.

Example The vector space $\mathcal{M}_{n\times m}$ has dimension $n \cdot m$. A natural basis consists of matrices with a single 1 and the other entries 0's.