Two.III Basis and Dimension

Linear Algebra

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Basis

Definition of basis

1.1 *Definition* A *basis* for a vector space is a sequence of vectors that is linearly independent and that spans the space.

Because a basis is a sequence, meaning that bases are different if they contain the same elements but in different orders, we denote it with angle brackets $\langle \vec{\beta}_1, \vec{\beta}_2, \ldots \rangle$.

Example This is a basis for $\mathbb{R}^2$.

$$\langle \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rangle$$

It is linearly independent.

$$c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \begin{array}{r} c_1 + c_2 = 0 \\ -c_1 + c_2 = 0 \end{array} \implies c_1 = 0,\ c_2 = 0$$

And it spans $\mathbb{R}^2$ since

$$c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \implies \begin{array}{r} c_1 + c_2 = x \\ -c_1 + c_2 = y \end{array}$$

has the solution $c_1 = (1/2)x - (1/2)y$ and $c_2 = (1/2)x + (1/2)y$.

1.13 *Definition* In a vector space with basis B the *representation of $\vec{v}$ with respect to B* is the column vector of the coefficients used to express $\vec{v}$ as a linear combination of the basis vectors:

$$\text{Rep}_B(\vec{v}) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

where $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ and $\vec{v} = c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \cdots + c_n\vec{\beta}_n$. The c's are the *coordinates of $\vec{v}$ with respect to B*.

Example In the vector space of linear polynomials $\mathcal{P}_1 = \{a + bx \mid a, b \in \mathbb{R}\}$ one basis is $B = \langle 1 + x, 1 - x \rangle$.

Check that is a basis by verifying that it is linearly independent

$$0 = c_1(1+x)+c_2(1-x) \implies 0 = c_1+c_2,\ 0 = c_1-c_2 \implies c_1 = c_2 = 0$$

and that it spans the space.

$$a + bx = c_1(1 + x) + c_2(1 - x) \implies c_1 = (a + b)/2,\ c_2 = (a - b)/2$$

Example This is a basis for $\mathcal{M}_{2\times 2}$.

$$\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \rangle$$

This is another one.

$$\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \rangle$$