Warm up

$M_{2\times 2}$ & $\mathbb{R}^4$ are the same

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \leftrightarrow \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

How do mathematicians make that precise?

with a map $f: M_{2\times 2} \to \mathbb{R}^4$

Definition

domain codomain

A function $f: V \to W$ is a homomorphism or linear map if it preserves $+$ & $\cdot$:

$$f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2)$$

$$f(r\vec{v}) = r f(\vec{v})$$

Lemma

f is a homomorphism if and only if

$$f(r_1\vec{v}_1 + r_2\vec{v}_2) = r_1 f(\vec{v}_1) + r_2 f(\vec{v}_2)$$

Lemma

If f is a homomorphism $f(\vec{0}) = \vec{0}$.

Example: $f: \mathbb{R}^3 \to \mathbb{R}^2$ is a homomorphism

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix}$$

Definition

A homomorphism $f: V \to W$ is an isomorphism

if it is

1) one-to-one: if $f(\vec{v}_1) = f(\vec{v}_2)$ then $\vec{v}_1 = \vec{v}_2$

2) onto: for every $\vec{w} \in W$, there is $\vec{v} \in V$ with $f(\vec{v}) = \vec{w}$

Def: If there exists an isomorphism $f: V \to W$, then V & W are isomorphic.

Example: $f: \mathcal{M}_{2\times 2} \to \mathbb{R}^4$ is an isomorphism

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

- it is a homomorphism
- it is one-to-one
- it is unto

Example: $f: \mathbb{R}^3 \to \mathbb{R}^2$ is <u>not</u> an isomorphism

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix}$$

It is not one-to-one:

$$f\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = f\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

but $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

Theorem

Dimension characterizes isomorphism, in other words, V & W are isomorphic if and only if they are the same dimension.

Example: $f: \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ 0 \end{pmatrix}$ is not an isomorphism

- it is not one-to-one

$$f\begin{pmatrix} 1 \\ 0 \end{pmatrix} = f\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- it is not unto: there is no $\begin{pmatrix} x \\ y \end{pmatrix}$ with $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

So even though $\mathbb{R}^2$ is isomorphic to itself, not all maps are isomorphisms.

Example: $f: \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+1 \\ y \end{pmatrix}$ is not a homomorphism

$$f\left(r_1\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + r_2\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) = f\begin{pmatrix} r_1x_1 + r_2x_2 \\ r_1y_1 + r_2y_2 \end{pmatrix} = \begin{pmatrix} r_1x_1 + r_2x_2 + 1 \\ r_1y_1 + r_2y_2 \end{pmatrix}$$

$$\not\parallel$$

but $$r_1 f\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + r_2 f\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = r_1\begin{pmatrix} x_1+1 \\ y_1 \end{pmatrix} + r_2\begin{pmatrix} x_2+1 \\ y_2 \end{pmatrix} = \begin{pmatrix} r_1x_1 + r_2x_2 + r_1 + r_2 \\ r_1y_1 + r_2y_2 \end{pmatrix}$$

Example: $f: \mathbb{R}^2 \to \mathbb{R}^2$ is an isomorphism

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+y \\ x-y \end{pmatrix}$$

It is a homomorphism:

$$f\left(r_1\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + r_2\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) = f\begin{pmatrix} r_1x_1+r_2x_2 \\ r_1y_1+r_2y_2 \end{pmatrix} = \begin{pmatrix} r_1x_1+r_2x_2+r_1y_1+r_2y_2 \\ r_1x_1+r_2x_2-r_1y_1-r_2y_2 \end{pmatrix}$$

$$= \begin{pmatrix} r_1(x_1+y_1)+r_2(x_2+y_2) \\ r_1(x_1-y_1)+r_2(x_2-y_2) \end{pmatrix} = r_1\begin{pmatrix} x_1+y_1 \\ x_1-y_1 \end{pmatrix} + r_2\begin{pmatrix} x_2+y_2 \\ x_2-y_2 \end{pmatrix}$$

$$= r_1 f\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + r_2 f\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

It is one-to-one

Let $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ be such that $f\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = f\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

i.e. $\begin{pmatrix} x_1+y_1 \\ x_1-y_1 \end{pmatrix} = \begin{pmatrix} x_2+y_2 \\ x_2-y_2 \end{pmatrix}$ i.e. $\begin{aligned} x_1+y_1 &= x_2+y_2 \\ x_1-y_1 &= x_2-y_2 \end{aligned}$

Then $2x_1 = (x_1+y_1)+(x_1-y_1) = (x_2+y_2)-(x_2-y_2) = 2x_2$

so $x_1 = x_2$

$y_1 = y_1 + x_1 - x_1 = y_2 + x_2 - x_2 = y_2$ so $y_1 = y_2$

if $f\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = f\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ then $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

It is unto

Let $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$. Find $\begin{pmatrix} x \\ y \end{pmatrix}$ such that $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$

Well $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x-y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$

so $x+y=a$ $\quad 2x = a+b \quad x = \frac{1}{2}(a+b)$
$x-y=b$ $\quad 2y = a-b \quad y = \frac{1}{2}(a-b)$

$\begin{pmatrix} \frac{1}{2}(a+b) \\ \frac{1}{2}(a-b) \end{pmatrix} \in \mathbb{R}^2$ and $f\begin{pmatrix} \frac{1}{2}(a+b) \\ \frac{1}{2}(a-b) \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$