So if $A = \begin{pmatrix} 2 & -1 & 3 & 1 & 0 & 1 \\ 3 & 0 & 1 & 1 & 4 & -1 \\ 4 & -2 & 6 & 2 & 0 & 2 \\ 1 & 0 & 3 & 0 & 0 & 2 \end{pmatrix}$

then the column space is all vectors in $\mathbb{R}^4$

of the form $\begin{pmatrix} \frac{1}{2}z \\ y \\ z \\ w \end{pmatrix}$

and the row space is all vectors in $\mathbb{R}^6$, say

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$ such that
$$x_3 - x_2 - \frac{5}{2}x_4 - \frac{3}{2}x_6 = 0$$
$$x_1 - 2x_2 - \frac{7}{2}x_4 - \frac{1}{2}x_6 = 0$$
$$x_5 - 4x_2 - 4x_4 = 0$$

Both spaces have dimension 3 but they are not the same space!

Another way to write row space: of the form

$$\begin{pmatrix} 5/2\,x_4 + 1/2\,x_5 + 1/2\,x_6 \\ -x_4 + 1/4\,x_5 \\ 3/2\,x_4 + 1/4\,x_5 + 3/2\,x_6 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

Notice:

ROW rank(A) = column rank (A)

This is always true!

Def: rank(A) = col rank(A) = ROW rank(A)

The rank of a matrix is great!

<u>Theorem</u>

Let $A \in M_{n \times m}$. The following are equivalent

1) rank(A) = r

2) the space $\{\vec{x} : A\vec{x} = \vec{0}\}$ has dimension $m - r$

$\leftarrow$ constants

system $a_{11}x_1 + \ldots + a_{1m}x_m = 0$

n = rank(A) + nullity(A)

r is how much you can make

nullity is how much you kill

n is total dimensions

Corollary

Let $A \in M_{n \times n}$ (square matrix) The following are equivalent:

1) $\text{rank}(A) = n$
2) A is nonsingular $A\vec{x} = \vec{0}$ has unique solution $\vec{x} = \vec{0}$

3) Rows are lin indep
4) cols are lin indep
5) for <u>any</u> vector $\vec{b}$, $A\vec{x} = \vec{b}$ <u>has</u> a <u>unique</u> solution

<u>Moral</u> Say I have $\mathbb{R}^4$ & two constraints $A = \left(\begin{smallmatrix} \text{———} \\ \text{———} \end{smallmatrix} \right)$

e.g. $a+b+c=0$
$a+b-c=0$ $\rightarrow$ $A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{pmatrix}$

nullity (A) is the dimension of $A \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Rank $A = 2$ means 2 constraints are <u>different</u>

$$A = \begin{pmatrix} 2 & -1 & 3 & 1 & 0 & 1 \\ 3 & 0 & 1 & 1 & 4 & -1 \\ 4 & -2 & 6 & 2 & 0 & 2 \\ 1 & 0 & 3 & 0 & 0 & 2 \end{pmatrix}$$

$\mathbb{R}^6$ 4 equations
but rank $A = 3$ so 3 constraints
so nullity $= 6 - 3 = 3$