Any Matrix Represents a Linear Map

The prior subsection shows how to start with a linear map and produce its matrix representation. What about the converse?

Example Fix a matrix

$$H = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

and also fix a domain and codomain, with bases.

$$\mathcal{E}_2 \subset \mathbb{R}^2 \quad \langle 1-x, 1+x \rangle \subset \mathcal{P}_1$$

Is there a linear map between the spaces associated with the matrix?

Consider $h\colon \mathbb{R}^2 \to \mathcal{P}_1$ defined by: for any domain vector $\vec{v}$, represent it with respect to the domain basis

$$\vec{v} = c_1\vec{e}_1 + c_2\vec{e}_2 \qquad \mathrm{Rep}_{E_2}(\vec{v}) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

multiply that representation by H

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 + 2c_2 \\ 3c_1 + 4c_2 \end{pmatrix}$$

and then call $h(\vec{v})$ the codomain vector represented by the result.

$$h(\vec{v}) = (c_1 + 2c_2) \cdot (1 - x) + (3c_1 + 4c_2) \cdot (1 + x)$$

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2.2 *Theorem* Any matrix represents a homomorphism between vector spaces of appropriate dimensions, with respect to any pair of bases.

The book has the proof.

Mechanics of Matrix Multiplication

3.8 *Definition* The *main diagonal* (or *principle diagonal* or *diagonal*) of a square matrix goes from the upper left to the lower right.

3.9 *Definition* An *identity matrix* is square and every entry is 0 except for 1's in the main diagonal.

$$I_{n \times n} = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ & \vdots & & \\ 0 & 0 & \ldots & 1 \end{pmatrix}$$

Taking the product with an identity matrix returns the multiplicand.

Example Multiplication by an identity from the left

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ -1 & 5 \end{pmatrix}$$

or from the right leaves the matrix unchanged.

$$\begin{pmatrix} 3 & 2 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ -1 & 5 \end{pmatrix}$$