Nov 2

Recall: If $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \in \mathcal{M}_{n \times m}$

is a matrix then its rows & columns are vectors

ROWS: $\vec{r}_1 = (a_{11}\ a_{12} \ldots a_{1m})^T$
$\vec{r}_2 = (a_{21} \ldots \quad a_{2m})^T \quad \in \mathbb{R}^m$
$\vdots$
$\vec{r}_n = (a_{n1} \ldots \quad a_{nm})^T$

columns $\vec{c}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}$ $\vec{c}_2 = \begin{pmatrix} a_{12} \\ \vdots \\ a_{n2} \end{pmatrix} \cdots$ $\vec{c}_m = \begin{pmatrix} a_{1m} \\ \vdots \\ a_{nm} \end{pmatrix}$

$\in \mathbb{R}^n$

The row space is the space spanned by the row vectors

The column space " by the column vectors

The row rank is the dim of the row space.

column rank dim of the column space.

Example

$$A = \begin{pmatrix} 2 & -1 & 3 & 1 & 0 & 1 \\ 3 & 0 & 1 & 1 & 4 & -1 \\ 4 & -2 & 6 & 2 & 0 & 2 \\ 1 & 0 & 3 & 0 & 0 & 2 \end{pmatrix}$$

column space: throw out extra columns by row reducing and throwing out free columns

$$\begin{pmatrix} 1 & 0 & 3 & 0 & 0 & 2 \\ 2 & -1 & 3 & 1 & 0 & 1 \\ 3 & 0 & 1 & 1 & 4 & -1 \\ 4 & -2 & 6 & 2 & 0 & 2 \end{pmatrix} \begin{matrix} \rho_2 - 2\rho_1 \\ \rho_3 - 3\rho_1 \\ \sim \\ \rho_4 - 4\rho_1 \end{matrix} \begin{pmatrix} 1 & 0 & 3 & 0 & 0 & 2 \\ 0 & -1 & -3 & 1 & 0 & -3 \\ 0 & 0 & -8 & 1 & 4 & -7 \\ 0 & -2 & -6 & 2 & 0 & -6 \end{pmatrix} \begin{matrix} -\rho_2 \\ -\rho_3 \\ \sim \\ -\frac{1}{2}\rho_4 \end{matrix} \begin{pmatrix} 1 & 0 & 3 & 0 & 0 & 2 \\ 0 & 1 & 3 & -1 & 0 & 3 \\ 0 & 0 & 8 & -1 & -4 & 7 \\ 0 & 1 & 3 & -1 & 0 & 3 \end{pmatrix}$$

$$\underset{\sim}{\rho_4 - \rho_2} \begin{pmatrix} 1 & 0 & 3 & 0 & 0 & 2 \\ 0 & 1 & 3 & -1 & 0 & 3 \\ 0 & 0 & 8 & -1 & -4 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\underbrace{}_{\text{leading}} \quad \underbrace{}_{\text{free}}$$

column space is

$$\text{Span}\left\{ \begin{pmatrix} 2 \\ 3 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 6 \\ 3 \end{pmatrix} \right\}$$

you can check but these would be linearly indep too

Just the last column:

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \sim \begin{pmatrix} w \\ x-2w \\ y-3w \\ z-4w \end{pmatrix} \sim \begin{pmatrix} w \\ 2w-x \\ 3w-y \\ 2w-\frac{1}{2}z \end{pmatrix} \sim \begin{pmatrix} w \\ 2w-x \\ 3w-y \\ \underbrace{x-\frac{1}{2}z}_{=0} \end{pmatrix}$$

$x = \frac{1}{2}z$

$$\begin{pmatrix} \cancel{\frac{1}{2}z} \\ y \\ z \\ w \end{pmatrix}$$ are in column space

Row space: compute column space of the transpose matrix

$$A^T = \begin{pmatrix} 2 & 3 & 4 & 1 & x_1 \\ -1 & 0 & -2 & 0 & x_2 \\ 3 & 1 & 6 & 3 & x_3 \\ 1 & 1 & 2 & 0 & x_4 \\ 0 & 4 & 0 & 0 & x_5 \\ 1 & -1 & 2 & 2 & x_6 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 2 & 2 & x_6 \\ 1 & 1 & 2 & 0 & x_4 \\ -1 & 0 & -2 & 0 & x_2 \\ 3 & 1 & 6 & 3 & x_3 \\ 2 & 3 & 4 & 1 & x_1 \\ 0 & 4 & 0 & 0 & x_5 \end{pmatrix} \begin{matrix} \rho_2-\rho_1 \\ \rho_3+\rho_1 \\ \sim \\ \rho_4-3\rho_1 \\ \rho_5-2\rho_1 \end{matrix}$$

$$\begin{pmatrix} 1 & -1 & 2 & 2 & x_6 \\ 0 & 2 & 0 & -2 & x_4-x_6 \\ 0 & -1 & 0 & +2 & x_2+x_6 \\ 0 & 4 & 0 & -3 & x_3-3x_6 \\ 0 & 5 & 0 & -3 & x_1-2x_6 \\ 0 & 4 & 0 & 0 & x_5 \end{pmatrix} \begin{matrix} \rho_3-2\rho_2 \\ \rho_4-4\rho_2 \\ \sim \\ \rho_5-5\rho_2 \\ \rho_6-4\rho_2 \end{matrix} \begin{pmatrix} 1 & -1 & 2 & 2 & x_6 \\ 0 & 1 & 0 & -2 & -x_2-x_6 \\ 0 & 0 & 0 & 2 & x_4+2x_2+x_6 \\ 0 & 0 & 0 & 5 & x_3+4x_2+x_6 \\ 0 & 0 & 0 & 7 & x_1+5x_2+3x_6 \\ 0 & 0 & 0 & 8 & x_5+4x_2+4x_6 \end{pmatrix}$$

free

so row space $=$ Span$\{\vec{r}_1^{\,T}, \vec{r}_2^{\,T}, \vec{r}_4^{\,T}\}$

Just bottom rows:

$$\begin{matrix} \\ \rho_4-\frac{5}{2}\rho_3 \\ \rho_5-\frac{7}{2}\rho_3 \\ \rho_6-4\rho_3 \end{matrix} \begin{pmatrix} 0 & 0 & 0 & 1 & \frac{1}{2}(x_4+2x_2+x_6) \\ 0 & 0 & 0 & 0 & x_3-x_2-\frac{5}{2}x_4-\frac{3}{2}x_6 \\ 0 & 0 & 0 & 0 & x_1-2x_2-\frac{7}{2}x_4-\frac{1}{2}x_6 \\ 0 & 0 & 0 & 0 & x_5-4x_2-4x_4 \end{pmatrix}$$