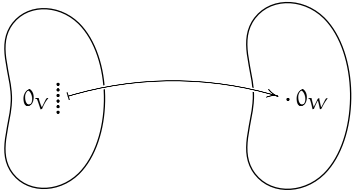
Range space and null space

2.11 *Definition* The ***null space*** or ***kernel*** of a linear map $h\colon V \to W$ is the inverse image of $\vec{0}_W$.

$$\mathscr{N}(h) = h^{-1}(\vec{0}_W) = \{\vec{v} \in V \mid h(\vec{v}) = \vec{0}_W\}$$

The dimension of the null space is the map's ***nullity***.



Note Strictly, the trivial subspace of the codomain is not $\vec{0}_W$, it is $\{\vec{0}_W\}$, and so we may think to write the nullspace as $h^{-1}(\{\vec{0}_W\})$. But we have defined the two sets $h^{-1}(\vec{w})$ and $h^{-1}(\{\vec{w}\})$ to be equal and the first is easier to write.

Range space

2.2 *Definition* The *range space* of a homomorphism $h\colon V \to W$ is

$$\mathscr{R}(h) = \{h(\vec{v}) \mid \vec{v} \in V\}$$

sometimes denoted $h(V)$. The dimension of the range space is the map's *rank*.

Example This map from $\mathcal{M}_{2\times 2}$ to $\mathbb{R}^2$ is linear.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \stackrel{h}{\longmapsto} \begin{pmatrix} a+b \\ 2a+2b \end{pmatrix}$$

The range space is a line through the origin

$$\{\begin{pmatrix} t \\ 2t \end{pmatrix} \mid t \in \mathbb{R}\}$$

(every member of that set is the image

$$\begin{pmatrix} t \\ 2t \end{pmatrix} = h(\begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix})$$

of a 2×2 matrix). The map's rank is 1.

Example The homomorphism $f\colon \mathcal{M}_{2\times 2} \to \mathbb{R}^2$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \stackrel{f}{\longmapsto} \begin{pmatrix} a+b \\ c+d \end{pmatrix}$$

has this null space

$$\begin{aligned} \mathscr{N}(f) &= \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a+b=0 \text{ and } c+d=0 \} \\ &= \{ \begin{pmatrix} -b & b \\ -d & d \end{pmatrix} \mid b, d \in \mathbb{R} \} \end{aligned}$$

and a nullity of 2.

Example The dilation function $d_3\colon \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} 3a \\ 3b \end{pmatrix}$$

has $\mathscr{N}(d_3) = \{\vec{0}\}$. A trivial space has an empty basis so d_3's nullity is 0.

Rank plus nullity

2.14 *Theorem* A linear map's rank plus its nullity equals the dimension of its domain.

The book contains the proof.

Example Consider this map $h\colon \mathbb{R}^3 \to \mathbb{R}$.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \overset{h}{\longmapsto} x/2 + y/5 + z$$

The null space is this plane.

$$\mathscr{N}(h) = h^{-1}(0) = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x/2 + y/5 + z = 0 \}$$

Other inverse image sets are also planes.

$$h^{-1}(1) = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x/2 + y/5 + z = 1 \} = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid z = 1 - x/2 - y/5 \}$$