MATH 124 Friday 9:40–10:30

Perkins 107

Start with Quiz 20

Examples of homomorphisms

(i) $f: \mathbb{R}^3 \to \mathbb{R}^3$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

- it is a homomorphism:

$$f\left(r_1 \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + r_2 \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \right) = f \begin{pmatrix} r_1 x_1 + r_2 x_2 \\ r_1 y_1 + r_2 y_2 \\ r_1 z_1 + r_2 z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$r_1 f \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + r_2 f \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = r_1 \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + r_2 \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \checkmark$$

- it is <u>not</u> one-to-one:

$$f \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = f \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} \quad \text{but} \quad \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \neq \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}$$

- it is <u>not</u> onto:

$\begin{pmatrix}1\\2\\3\end{pmatrix} \in \mathbb{R}^3$ but there is no $\begin{pmatrix}x\\y\\z\end{pmatrix}$ with $f\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix}$

(2) $f: \mathbb{R} \to \mathbb{R}^3$

$$x \mapsto \begin{pmatrix}x\\2x\\3x\end{pmatrix}$$

- it is a homomorphism

$$f(r_1 x_1 + r_2 x_2) = \begin{pmatrix} r_1 x_1 + r_2 x_2 \\ 2(r_1 x_1 + r_2 x_2) \\ 3(r_1 x_1 + r_2 x_2) \end{pmatrix} = \begin{pmatrix} r_1 x_1 + r_2 x_2 \\ 2r_1 x_1 + 2r_2 x_2 \\ 3r_1 x_1 + 3r_2 x_2 \end{pmatrix}$$

$$r_1 f(x_1) + r_2 f(x_2) = r_1 \begin{pmatrix} x_1 \\ 2x_1 \\ 3x_1 \end{pmatrix} + r_2 \begin{pmatrix} x_2 \\ 2x_2 \\ 3x_2 \end{pmatrix} = \begin{pmatrix} r_1 x_1 + r_2 x_2 \\ 2r_1 x_1 + 2r_2 x_2 \\ 3r_1 x_1 + 3r_2 x_2 \end{pmatrix}$$

Examples of isomorphisms

Recall: If V is a v.s., B is a basis for V,

$B = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$ and $\vec{v} \in V$

Then $\vec{v} = a_1\vec{b}_1 + a_2\vec{b}_2 + \ldots + a_n\vec{b}_n$

for some <u>unique</u> $a_1, a_2, \ldots, a_n$

We write $\operatorname{Rep}_B(\vec{v}) = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$

This is a source of isomorphisms

3 Because $\dim \mathcal{P}_2 = 3$ $\quad \mathcal{P}_2 \cong \mathbb{R}_3$

We can use the basis $B = \{1, x, x^2\}$ to give this isomorphism

$$f: \mathcal{P}_2 \longmapsto \mathbb{R}^3$$

$$a_0 + a_1x + a_2x^2 \mapsto \operatorname{Rep}_{\{1,x,x^2\}}(a_0 + a_1x + a_2x^2) = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$$

4 Consider the plane $V = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} s : t, s \in \mathbb{R} \right\}$

dimension of V is 2 so $V \cong \mathbb{R}_2$

$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis

Isomorphism:

$$f: V \to \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \mapsto \mathrm{Rep}_{\left\{\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}\right\}} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

$$f\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{because} \quad \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \cdot 1 + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \cdot 0$$

$$f\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{because} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \cdot 0 + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \cdot 1$$

What is $f\begin{pmatrix} 2 \\ 3 \\ 6 \\ 7 \end{pmatrix}$?

It is $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ where $\begin{pmatrix} 2 \\ 3 \\ 6 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} a_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} a_2$

i.e.

$$\begin{aligned} a_1 &= 2 \\ 2a_1 + a_2 &= 3 \\ 3a_1 &= 6 \\ 4a_1 + a_2 &= 7 \end{aligned}$$

if $a_1 = 2$ then

$2 \cdot 2 + a_2 = 3 \Rightarrow a_2 = -1$

so $f\begin{pmatrix}2\\3\\6\\7\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix} \in \mathbb{R}^2$

5 $f: \mathbb{R}^2 \to \mathbb{R}^2$ is an isomorphism

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x+y\\x-y\end{pmatrix}$$

• it is a homomorphism

$$f\left(r_1\begin{pmatrix}x_1\\y_1\end{pmatrix} + r_2\begin{pmatrix}x_2\\y_2\end{pmatrix}\right) = f\begin{pmatrix}r_1x_1+r_2x_2\\r_1y_1+r_2y_2\end{pmatrix} = \begin{pmatrix}r_1x_1+r_2x_2+r_1y_1+r_2y_2\\r_1x_1+r_2x_2-r_1y_1-r_2y_2\end{pmatrix}$$

$$r_1f\begin{pmatrix}x_1\\y_1\end{pmatrix} + r_2f\begin{pmatrix}x_2\\y_2\end{pmatrix} = r_1\begin{pmatrix}x_1+y_1\\x_1-y_1\end{pmatrix} + r_2\begin{pmatrix}x_2+y_2\\x_2-y_2\end{pmatrix} = \begin{pmatrix}r_1x_1+r_1y_1+r_2x_2+r_2y_2\\r_1x_1-r_1y_1+r_2x_2-r_2y_2\end{pmatrix}$$

• it is one-to-one

If $f\begin{pmatrix}x_1\\y_1\end{pmatrix} = f\begin{pmatrix}x_2\\y_2\end{pmatrix}$ then $\begin{pmatrix}x_1+y_1\\x_1-y_1\end{pmatrix} = \begin{pmatrix}x_2+y_2\\x_2-y_2\end{pmatrix}$

i.e. $x_1+y_1 = x_2+y_2$

$x_1-y_1 = x_2-y_2$

$$\left.\begin{array}{l}\text{add equations: } 2x_1 = 2x_2 \text{ so } x_1 = x_2\\ \text{subtract equations: } 2y_1 = 2y_2 \text{ so } y_1 = y_2\end{array}\right\} \text{Then } \begin{pmatrix}x_1\\y_1\end{pmatrix} = \begin{pmatrix}x_2\\y_2\end{pmatrix}$$

- it is onto

Fix $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$. We need to find $\begin{pmatrix} x \\ y \end{pmatrix}$ with $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$

i.e. $x+y=a$ add equations: $2x=a+b$

$x-y=b$ subtract equations: $2y=a-b$

$$f\begin{pmatrix} \frac{1}{2}(a+b) \\ \frac{1}{2}(a-b) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(a+b)+\frac{1}{2}(a-b) \\ \frac{1}{2}(a+b)-\frac{1}{2}(a-b) \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(a+b) \\ \frac{1}{2}(a-b) \end{pmatrix}$ works.

6 $f: \mathbb{R}^2 \to \mathcal{P}_1$

$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto a+(a+b)x$

- it is a homomorphism

$$f\left(r_1\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}+r_2\begin{pmatrix} a_2 \\ b_2 \end{pmatrix}\right) = f\begin{pmatrix} r_1a_1+r_2a_2 \\ r_1b_1+r_2b_2 \end{pmatrix} = (r_1a_1+r_2a_2) + (r_1a_1+r_2a_2+r_1b_1+r_2b_2)x$$

$$r_1f\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}+r_2f\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = r_1(a_1+(a_1+b_1)x)+r_2(a_2+(a_2+b_2)x)$$
$$= (r_1a_1+r_2a_2)+(r_1a_1+r_1b_1+r_2a_2+r_2b_2)x$$

- it is one-to-one

If $f\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = f\begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$ then $a_1 + (a_1 + b_1)x = a_2 + (a_2 + b_2)x$

then $a_1 = a_2$

$a_1 + b_1 = a_2 + b_2 \Rightarrow b_1 = b_2$

then $a_1 = a_2$ and $b_1 = b_2$

so $\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$

- it is onto

Take $c + dx \in P_1$. We need to find $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$

such that $f\begin{pmatrix} a \\ b \end{pmatrix} = c + dx$

i.e. $a + (a+b)x = c + dx$

Take $a = c$

$a + b = d$ so $b = d - a = d - c$

$$f\begin{pmatrix} c \\ d-c \end{pmatrix} = c + (c + (d-c))x = c + dx$$

so $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} c \\ d-c \end{pmatrix}$ works.