Homomorphisms and matrices

Convention: From now on, we will focus on homomorphisms $f: \mathbb{R}^m \to \mathbb{R}^n$.

Why? If $f: V \to W$

V is m-dimensional with basis $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$

W is n-dimensional with basis $\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n$

Then f must come from some $g: \mathbb{R}^m \to \mathbb{R}^n$

with

$$\begin{array}{ccccc} \vec{v} & V & \xrightarrow{f} & W & a_1\vec{w}_1 + \dots + a_n\vec{w}_n \\ \downarrow & \downarrow & & \uparrow & \uparrow \\ \mathrm{Rep}_{\{\vec{v}_1,\dots,\vec{v}_m\}}\vec{v} & \mathbb{R}^m & \xrightarrow{g} & \mathbb{R}^n & \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \end{array}$$

So we are not missing out

Fact: Any homomorphism f is determined by its action on a basis of the domain.

If V is the domain with basis $\vec{v}_1 \dots \vec{v}_m$

and $\vec{v} \in V$, then

$$f(\vec{v}) = f(a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_m\vec{v}_m)$$
$$= a_1 f(\vec{v}_1) + a_2 f(\vec{v}_2) + \dots + a_m f(\vec{v}_m)$$

Example: $f: \mathbb{R}^2 \to \mathbb{R}^2$

$$f\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} \quad f\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix}$$

then $f\begin{pmatrix}2\\5\end{pmatrix} = f\left(2\begin{pmatrix}1\\0\end{pmatrix} + 5\begin{pmatrix}0\\1\end{pmatrix}\right) = 2\begin{pmatrix}1\\1\end{pmatrix} + 5\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}7\\-3\end{pmatrix}$

$$f\begin{pmatrix}x\\y\end{pmatrix} = f\left(x\begin{pmatrix}1\\0\end{pmatrix} + y\begin{pmatrix}0\\1\end{pmatrix}\right) = x\begin{pmatrix}1\\1\end{pmatrix} + y\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}x+y\\x-y\end{pmatrix}$$

Example: $g: \mathbb{R}^2 \to \mathbb{R}^2$
$$\begin{pmatrix}a\\b\end{pmatrix} \mapsto \begin{pmatrix}a\\0\end{pmatrix}$$

$$g\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \quad g\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

Bookkeeping: For a basis $\vec{v}_1, \vec{v}_2 \dots \vec{v}_m$, write

the matrix $A = \begin{pmatrix} f(\vec{v}_1) & f(\vec{v}_2) & \dots & f(\vec{v}_m) \end{pmatrix}$

columns of A

Example: f is given by $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$\nearrow f\begin{pmatrix}1\\0\end{pmatrix}$ $\nwarrow f\begin{pmatrix}0\\1\end{pmatrix}$

g is given by $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$\nearrow g\begin{pmatrix}1\\0\end{pmatrix}$ $\nwarrow g\begin{pmatrix}0\\1\end{pmatrix}$

Example: $h: \mathbb{R}^2 \to \mathbb{R}$

$$\begin{pmatrix}a\\b\end{pmatrix} \mapsto 2a+3b$$

$h\begin{pmatrix}1\\0\end{pmatrix} = 2$ $h\begin{pmatrix}0\\1\end{pmatrix} = 3$ so h is given by $\begin{pmatrix} 2 & 3 \end{pmatrix}$

$\nearrow h\begin{pmatrix}1\\0\end{pmatrix}$ $\uparrow h\begin{pmatrix}0\\1\end{pmatrix}$

Fact: $f: \mathbb{R}^m \to \mathbb{R}^n$ will be given by a $n \times m$ matrix

Definition: If f is represented by A, we define the matrix-vector product

$$A \cdot \underset{\in \mathbb{R}^m}{\vec{x}} = \underset{\in \mathbb{R}^n}{f(\vec{x})}$$

This is a generalization of the dot product

Example:

$$\begin{pmatrix} 3 & 1 & 2 \\ 0 & -2 & 5 \end{pmatrix}\begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix} = \begin{pmatrix} 3\cdot 4 + 1\cdot(-1) + 2(-3) \\ 0\cdot 4 + (-2)(-1) + 5(-3) \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ -13 \end{pmatrix}$$

Fact: Every matrix is a homomorphism

$$\begin{pmatrix} 3 & 1 & 2 \\ 0 & -2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x + y + 2z \\ -2y + 5z \end{pmatrix}$$

$$f: \mathbb{R}^3 \to \mathbb{R}^2$$

Range space & null space

$f: \mathbb{R}^m \to \mathbb{R}^n$ has 2 important spaces associated to it

- the <u>range space</u>: everything in the image of f (everything "that gets hit" by f)

$$\left\{ \vec{w} \in \mathbb{R}^n \text{ such that there is } \vec{v} \in \mathbb{R}^m \text{ with } f(\vec{v}) = \vec{w} \right\} \subseteq \mathbb{R}^n$$

Its dimension is called the <u>rank</u> of f

- the <u>null space</u>: everything that gets sent to 0

$$\{ \vec{v} \in \mathbb{R}^m : f(\vec{v}) = 0 \}$$

Its dimension is called the <u>nullity</u> of f.

If f is given by the matrix A, then the range space is the column space of A.

Example: $f: \mathbb{R}^3 \to \mathbb{R}^2$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 3x + y + 2z \\ -2y + 5z \end{pmatrix}$$

Associated matrix: $A = \begin{pmatrix} 3 & 1 & 2 \\ 0 & -2 & 5 \end{pmatrix}$

$$\overset{-\frac{1}{2}\rho_2}{\sim} \begin{pmatrix} 3 & 1 & 2 \\ 0 & 1 & -5/2 \end{pmatrix} \overset{\rho_1 - \rho_2}{\sim} \begin{pmatrix} 3 & 0 & 9/2 \\ 0 & 1 & -5/2 \end{pmatrix} \overset{\frac{1}{3}\rho_1}{\sim} \begin{pmatrix} 1 & 0 & 3/2 \\ 0 & 1 & -5/2 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{leading variables}} \begin{matrix} 3/2 \\ -5/2 \end{matrix}$$

free variable

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3/2 \\ 5/2 \\ 1 \end{pmatrix} z$$

gives the null space

basis $\left\{ \begin{pmatrix} -3/2 \\ 5/2 \\ 1 \end{pmatrix} \right\}$

leading variables

↳ give the column space / range space

basis $\left\{ \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}$

The dimension of the domain gets totally "used up"
either it goes to the null space or to the range space

dim of domain = dim null + dim range

all variables = free + leading

Connection to before:

f is one-to-one if and only if its nullity is 0
(only $\vec{0}$ goes to $\vec{0}$)

Also f is onto if and only if its rank is equal to the dimension of the codomain or target space.

Composition / Matrix multiplication

Let f be represented by the matrix A

g " matrix B

If f and g have the same domain,

$f+g$ is represented by $A+B$ (matrix addition)

Also if $r \in \mathbb{R}$,

rf is represented by rA

Now suppose that $f: \mathbb{R}^m \to \mathbb{R}^n$
$g: \mathbb{R}^n \to \mathbb{R}^k$

then we can <u>compose</u> f and g:

$$g \circ f: \mathbb{R}^m \xrightarrow{f} \mathbb{R}^n \xrightarrow{g} \mathbb{R}^k$$

notice order! do right-most first

$$(g \circ f)(\vec{x}) = g(f(\vec{x}))$$

We <u>define</u> the product BA of 2 matrices

to be the matrix corresponding to $g \circ f$.

Here $A \in \mathcal{M}_{n\times m}$ $\quad B \in \mathcal{M}_{k\times n}$

and $BA \in \mathcal{M}_{k\times m}$

Matrix multiplication is only defined if the sizes match up appropriately

Matrix multiplication generalizes the matrix-vector product

Matrix multiplication is <u>not</u> commutative!

Example

$$\begin{pmatrix} 3 & 1 & 6 \\ 2 & 5 & 9 \end{pmatrix}\begin{pmatrix} 2 & 0 & 4 \\ 1 & -3 & 5 \\ 4 & 2 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 3\cdot2+1\cdot1+6\cdot4 & 3\cdot0+1(-3)+6\cdot2 & 3\cdot4+1\cdot5+6\cdot7 \\ 2\cdot2+5\cdot1+9\cdot4 & 2\cdot0+5(-3)+9\cdot2 & 2\cdot4+5\cdot5+9\cdot7 \end{pmatrix}$$

$$= \begin{pmatrix} 31 & 9 & 59 \\ 45 & 3 & 96 \end{pmatrix}$$

The matrix $I = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$ ones on diagonal zeros everywhere else

is the identity

Example: $\begin{pmatrix} 2 & 3 & 1 \\ 1 & 4 & 6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 4 & 6 \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 1 \\ 1 & 4 & 6 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 4 & 6 \end{pmatrix}$$

Inverses

Let $A \in M_{n \times n}$. If there is a matrix B with

$$BA = I$$

and $$AB = I$$

then A is <u>invertible</u> and B is A's <u>inverse</u>

$$B = A^{-1}$$

Fact: A is invertible if and only if its rank is n.

A is invertible if and only if its nullity is 0.

A is invertible if and only if the associated homomorphism f is an isomorphism.

A is invertible if and only if it is row equivalent to I, the identity matrix.

How to find A^{-1}

- write the big matrix $(A \mid I)$
- do Gauss-Jordan to get A in reduced echelon form, do everything to I too
- you will end with $(I \mid A^{-1})$

Example $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 5 & 6 & 0 & 0 & 0 & 1 \end{array}\right) \overset{\rho_3 - 5\rho_1}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & -4 & -15 & -5 & 0 & 1 \end{array}\right)$$

$$\overset{\rho_3 + 4\rho_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right)$$

$$\underset{\rho_1 - 3\rho_3}{\overset{\rho_2 - 4\rho_3}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 16 & -12 & -3 \\ 0 & 1 & 0 & 20 & -15 & -4 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right)$$

$$\overset{\rho_1 - 2\rho_1}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -24 & 18 & 5 \\ 0 & 1 & 0 & 20 & -15 & -4 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right)$$