Math 124: Fall 2016
Exam 2

NAME: SOLUTIONS

Time: **50 minutes**

For each problem, you **must** write down all of your work carefully and legibly to receive full credit. For each question, you **must** use theorems and/or mathematical reasoning to support your answer, as appropriate.

Failure to follow these instructions will constitute a breach of the UVM Code of Academic Integrity:

- You may not use a calculator or any notes or book during the exam.
- You may not access your cell phone during the exam for any reason; if you think that you will want to check the time please wear a watch.
- The work you present must be your own.
- Finally, you will more generally be bound by the UVM Code of Academic Integrity, which stipulates among other things that you may not communicate with anyone other than the instructor during the exam, or look at anyone else's solutions.

I understand and accept these instructions.

Signature: ______________________________

Problem	Value	Score
1	6	
2	6	
3	6	
4	6	
5	8	
6	8	
7	10	
TOTAL	50	

Problem 1 : (6 points) For the map $f\colon \mathcal{P}_1 \to \mathbb{R}^2$ given by

$$a + bx \mapsto \begin{pmatrix} a-b \\ b \end{pmatrix},$$

find the image of the following elements:

a) $3 - 2x$

$a=3$
$b=-2$

$$f(3-2x) = \begin{pmatrix} 3-(-2) \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$

b) $2 + 2x$

$a=2$
$b=2$

$$f(2+2x) = \begin{pmatrix} 2-2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

c) x

$a=0$
$b=1$

$$f(x) = \begin{pmatrix} 0-1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Problem 2 : (6 points)

a) (2 points) Give the definition of the word basis.

~~A~~ basis is a linearly independent spanning set.

b) (4 points) Is the set

$$\left\{ \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \right\}$$

a basis for the space $\mathbb{R}^3$?

Check if it spans $\mathbb{R}^3$: Is there a solution to the equation $a_1\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} + a_2\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + a_3\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

for any choice of $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$?

$$\left(\begin{array}{ccc|c} 0 & 1 & 1 & x \\ 2 & 1 & 3 & y \\ -1 & 1 & 0 & z \end{array}\right) \overset{\rho_1 \leftrightarrow -\rho_3}{\sim} \left(\begin{array}{ccc|c} 1 & -1 & 0 & -z \\ 2 & 1 & 3 & y \\ 0 & 1 & 1 & x \end{array}\right)$$

$$\overset{\rho_2 - 2\rho_1}{\sim} \left(\begin{array}{ccc|c} 1 & -1 & 0 & -z \\ 0 & 3 & 3 & y+2z \\ 0 & 1 & 1 & x \end{array}\right) \overset{\rho_2 - 3\rho_3}{\sim} \left(\begin{array}{ccc|c} 1 & -1 & 0 & -z \\ 0 & 0 & 0 & y+2z-3x \\ 0 & 1 & 1 & x \end{array}\right)$$

This is a contradiction if $y+2z-3x \neq 0$. This doesn't span $\mathbb{R}^3$, it is <u>not</u> a basis.

Problem 3 : (6 points) Let

$$\vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Represent $\vec{v}$ with respect to the basis

$$B = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}.$$

We want a_1 and a_2 with

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

solve:
$$\left(\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & 1 & 2 \end{array}\right) \overset{\rho_2 - \rho_1}{\sim} \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 2 & 1 \end{array}\right)$$

$$\overset{\frac{1}{2}\rho_2}{\sim} \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & 1/2 \end{array}\right) \overset{\rho_1 + \rho_2}{\sim} \left(\begin{array}{cc|c} 1 & 0 & 3/2 \\ 0 & 1 & 1/2 \end{array}\right)$$

$$a_1 = 3/2 \quad a_2 = 1/2$$

$$\mathrm{Rep}_{\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}$$

Problem 4 : (6 points) Is the vector

$$\vec{v} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

in the column space of the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 2 & 5 \end{pmatrix}?$$

Is there a_1, a_2 with

$$\begin{pmatrix} 1 \\ -3 \end{pmatrix} = a_1 \begin{pmatrix} 2 \\ 2 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ 5 \end{pmatrix} \quad ?$$

solve:

$$\left(\begin{array}{cc|c} 2 & 1 & 1 \\ 2 & 5 & -3 \end{array}\right) \overset{\rho_2 - \rho_1}{\sim} \left(\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & 4 & -4 \end{array}\right)$$

$$\overset{\frac{1}{4}\rho_2}{\sim} \left(\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & 1 & -1 \end{array}\right) \overset{\rho_1 - \rho_2}{\sim} \left(\begin{array}{cc|c} 2 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right)$$

$$\overset{\frac{1}{2}\rho_1}{\sim} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \end{array}\right)$$

Yes there is ($a_1 = 1$, $a_2 = -1$) so $\vec{v}$ is in the column space of A.

Problem 5 : (8 points) Consider the homogeneous system of linear equations

$$\begin{aligned} x_1 - 4x_2 + 3x_3 - x_4 &= 0 \\ 2x_1 - 8x_2 + 6x_3 - 2x_4 &= 0 \end{aligned}$$

What is the dimension of its solution set? Support your answer by giving a basis.

First we solve

$$\begin{pmatrix} 1 & -4 & 3 & -1 \\ 2 & -8 & 6 & -2 \end{pmatrix} \overset{\rho_2 - 2\rho_1}{\sim} \begin{pmatrix} 1 & -4 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

so $x_1 = 4x_2 - 3x_3 + x_4$

in vector form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} x_4$$

The set $\left\{ \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ spans the solution space

The set is also linearly independent because if

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix} a_1 + \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} a_2 + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} a_3,$$ this forces $a_1 = a_2 = a_3 = 0$

This space of solutions has a basis with 3 elements so it is of dimension 3.

Problem 6 : (8 points) What is the rank of the matrix

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 5 & 1 & 1 \\ 6 & 4 & 3 \end{pmatrix}?$$

Please support your answer by giving the basis of a space (specify which space that is) and arguing that you have found a basis.

We give a basis for the column space

$$\begin{pmatrix} 1 & 3 & 2 \\ 5 & 1 & 1 \\ 6 & 4 & 3 \end{pmatrix} \underset{\rho_3 - 6\rho_1}{\overset{\rho_2 - 5\rho_1}{\sim}} \begin{pmatrix} 1 & 3 & 2 \\ 0 & -14 & -9 \\ 0 & -14 & -9 \end{pmatrix} \overset{\rho_3 - \rho_2}{\sim} \begin{pmatrix} 1 & 3 & 2 \\ 0 & -14 & -9 \\ 0 & 0 & 0 \end{pmatrix}$$

leading free

The third column is superfluous

A basis for the column space is $\left\{ \begin{pmatrix} 1 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$

(our method is guaranteed to give a basis)

The rank of A is its column rank which is 2.

Problem 7 : (10 points)

a) (3 points) Give the definition of the word isomorphism.

An isomorphism is a homomorphism that is one-to-one and onto.

b) (7 points) Show that the subspace given by the x-axis of $\mathbb{R}^3$ is isomorphic to $\mathbb{R}$. (The x-axis are the points where $y = 0$ and $z = 0$.)
Please solve this problem by giving an explicit isomorphism between the x-axis of $\mathbb{R}^3$ and $\mathbb{R}$. Make sure to check that your isomorphism satisfies all of the conditions that you state in your definition in part a).

A basis for the x-axis is $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}$, so an isomorphism is given by

$$f: \text{x-axis} \to \mathbb{R}$$
$$\begin{pmatrix}x\\0\\0\end{pmatrix} \mapsto \mathrm{Rep}_{\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}}\begin{pmatrix}x\\0\\0\end{pmatrix} = x$$

1- This is a homomorphism:

$$f\left(r_1\begin{pmatrix}x_1\\0\\0\end{pmatrix}+r_2\begin{pmatrix}x_2\\0\\0\end{pmatrix}\right)=f\begin{pmatrix}r_1x_1+r_2x_2\\0\\0\end{pmatrix}=r_1x_1+r_2x_2$$

$$r_1 f\begin{pmatrix}x_1\\0\\0\end{pmatrix}+r_2 f\begin{pmatrix}x_2\\0\\0\end{pmatrix}=r_1x_1+r_2x_2$$

2- This is one-to-one:

If $f\begin{pmatrix}x_1\\0\\0\end{pmatrix}=f\begin{pmatrix}x_2\\0\\0\end{pmatrix}$ i.e. $x_1=x_2$, then $\begin{pmatrix}x_1\\0\\0\end{pmatrix}=\begin{pmatrix}x_2\\0\\0\end{pmatrix}$

Please see next page for "onto"

Please use this page if you need extra space for any problem. (On the problem page, be sure to let me know to look here, and label each problem clearly if you work on multiple problems here.)

3 - This is onto:

For any $x \in \mathbb{R}$, $\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}$ on the x-axis is

such that $f\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} = x$.