Math 124: Fall 2016
Practice for Exam 1

NAME: SOLUTIONS

Time: **1 hour and 15 minutes**

For each problem, you **must** write down all of your work carefully and legibly to receive full credit. For each question, you **must** use theorems and/or mathematical reasoning to support your answer, as appropriate.

Failure to follow these instructions will constitute a breach of the UVM Code of Academic Integrity:

- You may not use a calculator or any notes or book during the exam.
- You may not access your cell phone during the exam for any reason; if you think that you will want to check the time please wear a watch.
- The work you present must be your own.
- Finally, you will more generally be bound by the UVM Code of Academic Integrity, which stipulates among other things that you may not communicate with anyone other than the instructor during the exam, or look at anyone else's solutions.

I understand and accept these instructions.

Signature: ________________________________

Problem	Value	Score
1	15	
2	4	
3	6	
4	6	
5	5	
6	5	
7	9	
TOTAL	50	

Problem 1 : (15 points) Solve the following systems of linear equations. For full credit, check your answer when a solution exists.

a)

$$\begin{aligned} 2y + z &= 2 \\ 2x - y + z &= 0 \\ -2x - y &= -1 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 0 & 2 & 1 & 2 \\ 2 & -1 & 1 & 0 \\ -2 & -1 & 0 & -1 \end{array}\right) \overset{\rho_2 \leftrightarrow \rho_1}{\sim} \left(\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 0 & 2 & 1 & 2 \\ -2 & -1 & 0 & -1 \end{array}\right) \overset{\rho_3 + \rho_2}{\sim} \left(\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 0 & 2 & 1 & 2 \\ 0 & -2 & 1 & -1 \end{array}\right)$$

$$\overset{\rho_3 + \rho_2}{\sim} \left(\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{array}\right) \underset{\frac{1}{2}\rho_3}{\overset{\substack{\rho_1 - \frac{1}{2}\rho_3 \\ \rho_2 - \frac{1}{2}\rho_3}}{\sim}} \left(\begin{array}{ccc|c} 2 & -1 & 0 & -1/2 \\ 0 & 2 & 0 & 3/2 \\ 0 & 0 & 1 & 1/2 \end{array}\right)$$

$$\underset{\frac{1}{2}\rho_2}{\overset{\rho_1 + \frac{1}{2}\rho_2}{\sim}} \left(\begin{array}{ccc|c} 2 & 0 & 0 & 1/4 \\ 0 & 1 & 0 & 3/4 \\ 0 & 0 & 1 & 1/2 \end{array}\right) \overset{\frac{1}{2}\rho_1}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1/8 \\ 0 & 1 & 0 & 3/4 \\ 0 & 0 & 1 & 1/2 \end{array}\right) \qquad \begin{aligned} x &= \tfrac{1}{8} \\ y &= \tfrac{3}{4} \\ z &= \tfrac{1}{2} \end{aligned}$$

Check: $2\left(\frac{3}{4}\right) + \frac{1}{2} = \frac{3}{2} + \frac{1}{2} = 2$ ✓

$2\left(\frac{1}{8}\right) - \frac{3}{4} + \frac{1}{2} = \frac{1}{4} - \frac{3}{4} + \frac{1}{2} = -\frac{1}{2} + \frac{1}{2} = 0$ ✓

$-2\left(\frac{1}{8}\right) - \frac{3}{4} = -\frac{1}{4} - \frac{3}{4} = -1$ ✓

b)

$$\begin{aligned} x - y + z &= 0 \\ y + w &= 0 \\ 3x - 2y + 3z + w &= 0 \\ -y - w &= 0 \end{aligned}$$

$$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 3 & -2 & 3 & 1 \\ 0 & -1 & 0 & -1 \end{pmatrix} \overset{\rho_3 - 3\rho_1}{\sim} \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & -1 \end{pmatrix} \underset{\rho_4 + \rho_2}{\overset{\rho_3 - \rho_2}{\sim}} \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\overset{\rho_1 + \rho_2}{\sim} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

z and w are free

$x = -z - w$

$y = -w$

Solution: $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} z + \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix} w$

Check: It is enough to check the 2 vectors that span the solution space.

$\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$-1 + 0 - 1 = 0$ ✓

$0 + 0 = 0$ ✓

$3(-1) - 0 + 3(1) = 0$ ✓

$-0 - 0 = 0$ ✓

$\begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$

$-1 - (-1) + 0 = 0$ ✓

$-1 + 1 = 0$ ✓

$3(-1) - 2(-1) + 0 + 1 = 0$ ✓

$-(-1) - 1 = 0$ ✓

c)

$$\begin{aligned} x + y + 2z &= 0 \\ 2x - y + z &= 1 \\ 4x + y + 5z &= -1 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 2 & -1 & 1 & 1 \\ 4 & 1 & 5 & -1 \end{array}\right) \underset{\rho_3 - 4\rho_1}{\overset{\rho_2 - 2\rho_1}{\sim}} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -3 & -3 & 1 \\ 0 & -3 & -3 & -1 \end{array}\right)$$

$$\overset{\rho_3 - \rho_2}{\sim} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -3 & -3 & 1 \\ 0 & 0 & 0 & -2 \end{array}\right)$$

The last row is a contradiction: $0 \neq -2$

Problem 2 : (4 points) Write down the 4×4 matrix whose $a_{i,j}$ entry is $(-1)^{i+j}$.

$$\begin{bmatrix} (-1)^{1+1} & (-1)^{1+2} & (-1)^{1+3} & (-1)^{1+4} \\ (-1)^{2+1} & (-1)^{2+2} & (-1)^{2+3} & (-1)^{2+4} \\ (-1)^{3+1} & (-1)^{3+2} & (-1)^{3+3} & (-1)^{3+4} \\ (-1)^{4+1} & (-1)^{4+2} & (-1)^{4+3} & (-1)^{4+4} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}$$

Problem 3 : (6 points) Are the following two matrices row equivalent?

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 2 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 3 & -3 \\ 1 & 1 & 5 \end{pmatrix}$$

We will give the reduced echelon form, which is unique, and compare. If the reduced echelon forms are the same then the matrices are row equivalent.

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 2 & 2 \end{pmatrix} \overset{\rho_2+\rho_1}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \end{pmatrix} \overset{\frac{1}{3}\rho_2}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \overset{\rho_1-\rho_2}{\sim} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

all done

$$\begin{pmatrix} 0 & 3 & -3 \\ 1 & 1 & 5 \end{pmatrix} \overset{\rho_1\leftrightarrow\rho_2}{\sim} \begin{pmatrix} 1 & 1 & 5 \\ 0 & 3 & -3 \end{pmatrix} \overset{\frac{1}{3}\rho_2}{\sim} \begin{pmatrix} 1 & 1 & 5 \\ 0 & 1 & -1 \end{pmatrix} \overset{\rho_1-\rho_2}{\sim} \begin{pmatrix} 1 & 0 & 6 \\ 0 & 1 & -1 \end{pmatrix}$$

all done

These are not the same so the 2 matrices are not row equivalent.

Problem 4 : (6 points) Consider the following subsets of $\mathcal{P}_3$, the set of polynomials of degree less than or equal to two. Decide if each of the subsets is linearly independent or linearly dependent.

a) $\{-1, 4+x^2\}$

What are the a_1 and a_2 such that

$$a_1(-1) + a_2(4+x^2) = 0 \quad ?$$

Simplify: $(4a_2 - a_1) + a_2 x^2 = 0$

A polynomial is zero only if all its coefficients (the constant term, the number in front of x, the number in front of x^2, etc) are zero.

So to be zero we need
$$4a_2 - a_1 = 0$$
$$a_2 = 0$$

$$\begin{pmatrix} 4 & -1 \\ 0 & 1 \end{pmatrix} \overset{\rho_1+\rho_2}{\sim} \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \overset{\frac{1}{4}\rho_1}{\sim} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The only solution is $a_1 = 0,\ a_2 = 0$

So the vectors are linearly independent.

b) $\{-x^2, 1+4x^2, 4\}$

What are the a_1, a_2, a_3 with

$$a_1(-x^2)+a_2(1+4x^2)+a_3 \cdot 4 = 0 \;?$$

Simplify: $(a_2+4a_3)+(4a_2-a_1)x^2=0$

Again set each coefficient to be zero, to get the zero polynomial:

$$\begin{aligned} a_2+4a_3 &= 0 \\ 4a_2-a_1 &= 0 \end{aligned}$$

$$\begin{pmatrix} 0 & 1 & 4 \\ -1 & 4 & 0 \end{pmatrix} \overset{\rho_2 \leftrightarrow -\rho_1}{\sim} \begin{pmatrix} 1 & -4 & 0 \\ 0 & 1 & 4 \end{pmatrix} \overset{\rho_1+4\rho_2}{\sim} \begin{pmatrix} 1 & 0 & 16 \\ 0 & 1 & 4 \end{pmatrix}$$

a_1 and a_2 are leading variables but a_3 is free. Therefore there are infinitely many solutions are the vectors are linearly dependent.

Problem 5 : (5 points) Let S be the set of all vectors in $\mathbb{R}^3$ that are perpendicular to the vector

$$\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}.$$

Show that the set S is a vector subspace of $\mathbb{R}^3$.

Let $\vec{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ such that $\vec{u} \cdot \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = x + 3y - z = 0$

The vectors perpendicular to $\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$ are all in $\mathbb{R}^3$ which is itself a vector space, so it suffices to check the subspace condition.

Let $\vec{u}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\vec{u}_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$ both be perpendicular to $\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$

i.e. $x_1 + 3y_1 - z_1 = 0$ and $x_2 + 3y_2 - z_2 = 0$

Let r_1 and r_2 be 2 real numbers. Then

$r_1 \vec{u}_1 + r_2 \vec{u}_2 = \begin{pmatrix} r_1 x_1 + r_2 x_2 \\ r_1 y_1 + r_2 y_2 \\ r_1 z_1 + r_2 z_2 \end{pmatrix}$. Is that also perpendicular to $\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$?

We compute $x + 3y + z$ and see if it is 0.

$$(r_1x_1+r_2x_2)+3(r_1y_1+r_2y_2)+(r_1z_1+r_2z_2)$$

$$=(r_1x_1+3r_1y_1+r_1z_1)+(r_2x_2+3r_2y_2+r_2z_2)$$

$$=r_1(x_1+3y_1+z_1)+r_2(x_2+3y_2+z_2)$$

$$=r_1\cdot 0+r_2\cdot 0=0$$

Yes, the linear combination is also perpendicular to $\begin{pmatrix}1\\3\\-1\end{pmatrix}$, so the set of vectors perpendicular to $\begin{pmatrix}1\\3\\-1\end{pmatrix}$ does satisfy the subspace property and is a vector subspace

Problem 6 : (5 points) Consider the following subset of $\mathbb{R}^2$:

$$\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}.$$

Is every vector in $\mathbb{R}^2$ in the span of this set?

we want to know; if x and y are any numbers, is there a_1 and a_2 such that

$$\begin{pmatrix} x \\ y \end{pmatrix} = a_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad ?$$

I.e. can every $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ be built using only $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$?

This gives us the equations
$$2a_1 + 3a_2 = x$$
$$a_1 + a_2 = y$$

$$\left(\begin{array}{cc|c} 2 & 3 & x \\ 1 & 1 & y \end{array}\right) \overset{\rho_1 \leftrightarrow \rho_2}{\sim} \left(\begin{array}{cc|c} 1 & 1 & y \\ 2 & 3 & x \end{array}\right) \overset{\rho_2 - 2\rho_1}{\sim} \left(\begin{array}{cc|c} 1 & 1 & y \\ 0 & 1 & x-2y \end{array}\right)$$

$$\overset{\rho_1 - \rho_2}{\sim} \left(\begin{array}{cc|c} 1 & 0 & 3y-x \\ 0 & 1 & x-2y \end{array}\right)$$

Yes! we just need to take $a_1 = 3y - x$
$a_2 = x - 2y$

For example: $\begin{pmatrix} -1 \\ -1 \end{pmatrix} = -2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ $\quad$ $a_1 = -3 + 1 = -2$
$a_2 = -1 + 2 = 1$

Problem 7 : (9 points) Consider the set of real numbers $\mathbb{R}$ with vector addition given by

$$x \oplus y = x + y + 7,$$

where $\oplus$ is the new vector addition and $+$ is the usual addition of real numbers, and scalar multiplication

$$r \otimes x = rx + 7(r - 1),$$

where again all operations on the right are the usual multiplication, addition and subtraction on real numbers.
This is a vector space.

a) Show that this vector addition is commutative.

Is $x \oplus y = y \oplus x$?

$x \oplus y = x + y + 7 = y + x + 7 = y \oplus x$ yes!

$\uparrow$
we know normal addition is commutative

b) What is the zero vector in this vector space?

We need to find a special number z such that $x \oplus z = x$ for all $x \in \mathbb{R}$

We just solve $x \oplus z = x$ for z:

$x \oplus z = x + z + 7 = x$

$z + 7 = 0$

$z = -7$

The zero vector here is the number -7.

For your convenience, here are the operations in this vector space again: Addition is given by

$$x \oplus y = x + y + 7,$$

and scalar multiplication is

$$r \otimes x = rx + 7(r - 1).$$

c) Show that addition of scalars distributes over scalar multiplication, i.e. that $(r+s) \otimes x = r \otimes x \oplus s \otimes x$.

This is a case where it might help to work out both ends of the chain of equality:

$$(r+s) \otimes x = (r+s)x + 7(r+s-1)$$

$$\begin{aligned} r \otimes x \oplus s \otimes x &= (rx + 7(r-1)) \oplus (sx + 7(s-1)) \\ &= rx + 7(r-1) + sx + 7(s-1) + 7 \\ &= rx + sx + 7(r-1) + 7(s-1) + 7 \\ &= (r+s)x + 7r + 7s - 7 - 7 + 7 \\ &= (r+s)x + 7(r+s-1) \\ &= (r+s) \otimes x \end{aligned}$$

Bonus! We check all the axioms!

① $x \oplus y$ is another real number

Yes, if $x, y \in \mathbb{R}$, then $x+y+7 \in \mathbb{R}$ also

② $\oplus$ is commutative: done

③ $\oplus$ is associative

$$\begin{aligned}(x \oplus y) \oplus z &= (x+y+7) \oplus z \\ &= (x+y+7) + z + 7 \\ &= x + (y+z+7) + 7 \\ &= x \oplus (y+z+7) \\ &= x \oplus (y \oplus z)\end{aligned}$$

④ There is a zero vector: yes, it is -7

⑤ Every vector has an additive inverse

For each $x \in \mathbb{R}$ I need to give some

$y \in \mathbb{R}$ with $x \oplus y = -7$

So I solve $x \oplus y = -7$ for y:

$$x \oplus y = x + y + 7 = -7$$

$$y = -x - 14$$

So for $x \in \mathbb{R}$, the additive inverse is $-x-14 \in \mathbb{R}$

see Bonus Bonus for more!

⑥ $r \otimes x$ is another real number

Yes, if $r, x \in \mathbb{R}$, then $rx + 7(r-1) \in \mathbb{R}$ also

⑦ 1st distributivity: $(r+s) \otimes x = (r \otimes x) \oplus (s \otimes x)$

done

⑧ 2nd distributivity: $r \otimes (x \oplus y) = (r \otimes x) \oplus (r \otimes y)$

Again we start both sides:

$$r \otimes (x \oplus y) = r \otimes (x + y + 7) = r(x + y + 7) + 7(r-1)$$
$$= r(x+y) + 14r - 7$$

$$(r\otimes x)\oplus(r\otimes y) = (rx+7(r-1))\oplus(ry+7(r-1))$$

$$= rx+7(r-1)+ry+7(r-1)+7$$

$$= rx+ry+14(r-1)+7$$

$$= r(x+y)+14r-7$$

$$= r(x+y+7)+7(r-1)$$

$$= r\otimes(x\oplus y)$$

(9) Associativity of multiplication:

$$r\otimes(s\otimes x) = (rs)\otimes x$$

$$r\otimes(s\otimes x) = r\otimes(sx+7(s-1))$$

$$= rsx+7r(s-1)+7(r-1)$$

$$= rsx+7rs-7r+7r-7$$

$$= rsx+7(rs-1)$$

$$= (rs)\otimes x$$

(10) $1\otimes x = x$: $1\otimes x = x+7(1-1) = x$

Bonus Bonus: What is the additive inverse for?

It allows us to solve equations.

Say I want to know x such that $x \oplus 2 = 3$

Sadly, it's not $x=1$: $1 \oplus 2 = 1+2+7 = 10 \neq 3$

This is because here $\ominus 2$ is not -2, it is

$-2-14 = -16$

Then $x \oplus 2 \oplus -16 = 3 \oplus -16$

"add" -16 on both sides

On the left we get $x \oplus (2 \oplus -16) = x \oplus (2-16+7)$

$= x \oplus -7$

$= x - 7 + 7 = x$

On the right we get $3 \oplus -16 = 3 - 16 + 7 = -6$

so $x = -6$

(check: $-6 \oplus 2 = -6 + 2 + 7 = 3$)