Eigenvectors and eigenvalues

Definition: A square matrix $A \in M_{n \times n}$ has <u>eigenvalue</u> λ associated with the nonzero <u>eigenvector</u> $\vec{v}$ if

$$A\vec{v} = \lambda \vec{v}$$

Example: If $A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$,

A has eigenvalue $\lambda = 2$ with eigenvector $\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ because

$$A\vec{v} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2\vec{v}$$

A also has eigenvalue $\lambda = 0$ with eigenvector $\vec{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ because

$$A\vec{v} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0\vec{v}$$

The vector $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is not an eigenvector:

$$A\vec{v} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \leftarrow \text{not a multiple of } \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

How to compute the eigenvalues of a matrix A:

Idea: $B\vec{v} = \vec{0}$ has a solution $\vec{v} \neq \vec{0}$ if and only if $\det B = 0$

Why?

If $\det B \neq 0$, B^{-1} exists. Then multiplying both sides of $B\vec{v} = \vec{0}$ by B^{-1} gives

$$B^{-1}B\vec{v} = B^{-1}\vec{0}$$
$$\vec{v} = \vec{0}$$

If $\det B = 0$, using the row reduction technique to compute determinants that we didn't cover, we can show that the echelon form of B has (at least) one zero row, so (at least) one free variable, so nullity at least one. Therefore the set of vectors $\vec{v}$ with $B\vec{v} = \vec{0}$ has dimension at least one and contains a non zero vector.

Now if A has eigenvalue λ with eigenvector $\vec{v}$, then

$$A\vec{v} = \lambda\vec{v}$$
$$A\vec{v} - \lambda\vec{v} = \vec{0}$$
$$\underbrace{(A-\lambda I)}_{\text{matrix } B \text{ from before}}\vec{v} = \vec{0}$$

where I is the $n \times n$ identity matrix

Theorem
The eigenvalues of A are exactly the values of λ such that $\det(A-\lambda I)=0$.

In practice, we start by treating λ as a variable, and compute the polynomial

$$p(\lambda) = \det(A-\lambda I)$$

Definition
$p(\lambda)=\det(A-\lambda I)$ is the <u>characteristic</u> polynomial of A.

The roots of $p(\lambda)$ are the eigenvalues.

Example: $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

$$A - \lambda I = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix}$$

$$\begin{aligned} p(\lambda) = \det(A - \lambda I) &= (3-\lambda)(3-\lambda) - 1 \\ &= 9 - 6\lambda + \lambda^2 - 1 \\ &= \lambda^2 - 6\lambda + 8 = (\lambda - 2)(\lambda - 4) \end{aligned}$$

A has characteristic polynomial $p(\lambda) = \lambda^2 - 6\lambda + 8$

and eigenvalues $\lambda_1 = 2$, $\lambda_2 = 4$.

How to find the eigenvectors:

We already know how!

- pick a specific eigenvalue
- write $B = A - \lambda I$
- solve $B\vec{v} = \vec{0}$ ← null space / solution set of a homogeneous system

Example: $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ eigenvalues
$\lambda_1 = 2$, $\lambda_2 = 4$

$\underline{\lambda_1 = 2}$ $B = A - 2I = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Solve $B\vec{v}_1 = \vec{0}$:

$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ so $x + y = 0$ $x = -y$

$\vec{v}_1 = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} y$

<u>Any</u> choice of y gives an eigenvector:

$y = 1$ $\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $A\vec{v}_1 = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} = 2\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

or

$y = -3$ $\vec{v}_1 = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$ $A\vec{v}_1 = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 3 \\ -3 \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \end{pmatrix} = 2\begin{pmatrix} 3 \\ -3 \end{pmatrix}$

$\underline{\lambda_2 = 4}$ $B = A - 4I = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$

Solve $B\vec{v}_2 = \vec{0}$:

$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$ so $-x + y = 0$ $x = y$

$\vec{v}_2 = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} y$

Again any choice of y gives an eigenvector
usually we pick $y = 1$ because it's convenient

$y=1$ $\vec{v}_2 = \begin{pmatrix}1\\1\end{pmatrix}$ $A\vec{v}_2 = \begin{pmatrix}3 & 1\\1 & 3\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}4\\4\end{pmatrix} = 4\begin{pmatrix}1\\1\end{pmatrix}$.

Definition

The set

$\{ \vec{v} : \vec{v}$ is an eigenvector with eigenvalue $\lambda \}$

when λ is fixed, is called the <u>eigenspace of A with eigenvalue λ</u>.

An eigenspace is a vector space because it is the solution set of a homogeneous system of equations (which is a vector space)

Theorem

If we join the bases of any finite number of eigenspaces into one set, that set will still be linearly independent.

Example $A = \begin{bmatrix} 1 & 5 & 0 \\ 1 & 5 & 0 \\ 1 & -1 & 6 \end{bmatrix}$

Find the eigenvalues and a basis for each eigenspace

$$p(\lambda) = \det(A - \lambda I)$$

$$= \begin{vmatrix} 1-\lambda & 5 & 0 \\ 1 & 5-\lambda & 0 \\ 1 & -1 & 6-\lambda \end{vmatrix}$$

expand along this column

$$= (-1)^{3+3}(6-\lambda)\begin{vmatrix} 1-\lambda & 5 \\ 1 & 5-\lambda \end{vmatrix}$$

$$= (6-\lambda)\left((1-\lambda)(5-\lambda) - 5 \right)$$

$$= (6-\lambda)(5 - 6\lambda + \lambda^2 - 5)$$

$$= (6-\lambda)(\lambda^2 - 6\lambda) = (6-\lambda)\lambda(\lambda-6)$$

There are 2 eigenvalues: $\lambda_1 = 0$, $\lambda_2 = 6$ (twice)

<u>$\lambda_1 = 0$</u> Solve $A\vec{v}_1 = \vec{0}$:

$$\begin{bmatrix} 1 & 5 & 0 \\ 1 & 5 & 0 \\ 1 & -1 & 6 \end{bmatrix} \underset{\rho_3-\rho_1}{\overset{\rho_2-\rho_1}{\sim}} \begin{bmatrix} 1 & 5 & 0 \\ 0 & 0 & 0 \\ 0 & -6 & 6 \end{bmatrix} \overset{\rho_2 \leftrightarrow -\frac{1}{6}\rho_3}{\sim} \begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\overset{\rho_1 - 5\rho_2}{\sim} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x + 5z = 0$
$y - z = 0$

$x = -5z$
$y = z$
$z = z$

$$\vec{v}_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix} z \qquad \vec{v}_1 = \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix}$$ (any choice of z is ok)

$\left\{ \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix} \right\}$ is a basis for the eigenspace with eigenvalue 0.

$\underline{\lambda_2 = 6}$ Solve $(A - 6I) = 0$

$$A - 6I = \begin{bmatrix} -5 & 5 & 0 \\ 1 & -1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \underset{\sim}{\rho_1 \leftrightarrow \rho_2} \begin{bmatrix} 1 & -1 & 0 \\ -5 & 5 & 0 \\ 1 & -1 & 0 \end{bmatrix} \underset{\rho_3 - \rho_1}{\overset{\rho_2 + 5\rho_1}{\sim}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x - y = 0$ so $x = y$, y & z are free

$$\vec{v}_2 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} y + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} z$$

$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for the eigenspace with eigenvalue 6

Remark: Sometimes a value of λ will be a factor of $p(\lambda)$ with multiplicity bigger than 1, but the dimension of the eigenspace will be smaller. However,

$$\text{dimension of eigenspace} \leq \text{multiplicity of eigenvalue as a root of } p(\lambda)$$

always

In our example, we can join the basis for the eigenspace associated to $\lambda_1 = 0$ and the basis for the eigenspace associated to $\lambda_2 = 6$, and the set is still linearly independent:

$$\left\{ \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Remark: Since these are 3 linearly independent vectors in $\mathbb{R}^3$, this is a basis for $\mathbb{R}^3$.

Definition

If $A \in M_{n \times n}$ and by putting together a basis for each eigenspace we get n linearly independent vectors, then A is <u>diagonalizable</u>.

Example of eigenvalue with multiplicity 2 but the eigenspace has dimension 1

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

$$\det(A - \lambda I) = \det\begin{pmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{pmatrix} = (3-\lambda)^2$$

$\lambda = 3$ is the only eigenvalue

Solve $(A - 3I) = 0$

$$A - 3I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$y = 0$
x is free

$$\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} x$$

A basis for the eigenspace is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.