Eigenvalues and Eigenvectors

Eigenvalues and eigenvectors

3.1 *Definition* A transformation $t\colon V \to V$ has a scalar *eigenvalue* λ if there is a nonzero *eigenvector* $\vec{\zeta} \in V$ such that $t(\vec{\zeta}) = \lambda \cdot \vec{\zeta}$.

3.5 *Definition* A square matrix T has a scalar *eigenvalue* λ associated with the nonzero *eigenvector* $\vec{\zeta}$ if $T\vec{\zeta} = \lambda \cdot \vec{\zeta}$.

Example The matrix

$$D = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$$

has an eigenvalue $\lambda_1 = 4$ and a second eigenvalue $\lambda_2 = 2$. The first is true because an associated eigenvector is $\vec{e_1}$

$$\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 4 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and similarly for the second an associated eigenvector is e_2.

$$\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Thinking of the matrix as representing a transformation of the plane, the transformation acts on those vectors in a particularly simple way, by rescaling.

Not every vector is simply rescaled.

$$\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \neq x \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Computing eigenvalues and eigenvectors

Example We will find the eigenvalues and associated eigenvectors of this matrix.

$$T = \begin{pmatrix} 0 & 5 & 7 \\ -2 & 7 & 7 \\ -1 & 1 & 4 \end{pmatrix}$$

We want to find scalars x such that $T\vec{\zeta} = x\vec{\zeta}$ for some nonzero $\vec{\zeta}$. Bring the terms to the left side.

$$\begin{pmatrix} 0 & 5 & 7 \\ -2 & 7 & 7 \\ -1 & 1 & 4 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} - x \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and factor.

$$\begin{pmatrix} 0-x & 5 & 7 \\ -2 & 7-x & 7 \\ -1 & 1 & 4-x \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad (*)$$

This homogeneous system has nonzero solutions if and only if the matrix is singular, that is, has a determinant of zero.

Some computation gives the determinant and its factors.

$$\begin{aligned} 0 &= \begin{vmatrix} 0-x & 5 & 7 \\ -2 & 7-x & 7 \\ -1 & 1 & 4-x \end{vmatrix} \\ &= x^3 - 11x^2 + 38x - 40 = (x-5)(x-4)(x-2) \end{aligned}$$

So the eigenvalues are $\lambda_1 = 5$, $\lambda_2 = 4$, and $\lambda_3 = 2$.

To find the eigenvectors associated with the eigenvalue of 5 specialize equation $(*)$ for $x = 5$.

$$\begin{pmatrix} -5 & 5 & 7 \\ -2 & 2 & 7 \\ -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Gauss's Method gives this solution set; its nonzero elements are the eigenvectors.

$$V_5 = \{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} z_2 \mid z_2 \in \mathbb{C} \}$$

Similarly, to find the eigenvectors associated with the eigenvalue of 4 specialize equation $(*)$ for $x = 4$.

$$\begin{pmatrix} -4 & 5 & 7 \\ -2 & 3 & 7 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Gauss's Method gives this.

$$V_4 = \{ \begin{pmatrix} -7 \\ -7 \\ 1 \end{pmatrix} z_3 \mid z_3 \in \mathbb{C} \}$$

Specializing $(*)$ for $x = 2$

$$\begin{pmatrix} -2 & 5 & 7 \\ -2 & 5 & 7 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

gives this.

$$V_2 = \{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} z_3 \mid z_3 \in \mathbb{C} \}$$

Example To find the eigenvalues and associated eigenvectors for the matrix

$$T = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

start with this equation.

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = x \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \quad \Longrightarrow \quad \begin{pmatrix} 3-x & 1 \\ 1 & 3-x \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (*)$$

That system has a nontrivial solution if this determinant is nonzero.

$$\begin{vmatrix} 3-x & 1 \\ 1 & 3-x \end{vmatrix} = x^2 - 6x + 8 = (x-2)(x-4)$$

First take the $x = 2$ version of $(*)$.

$$\begin{aligned} 1 \cdot b_1 + \phantom{1 \cdot{}} b_2 &= 0 \\ b_1 + 1 \cdot b_2 &= 0 \end{aligned} \quad \Longrightarrow \quad V_2 = \{ \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \mid b_1 = -b_2 \text{ where } b_2 \in \mathbb{C} \}$$

Solving the second system is just as easy.

$$\begin{aligned} -1 \cdot b_1 + \phantom{1 \cdot{}} b_2 &= 0 \\ b_1 - 1 \cdot b_2 &= 0 \end{aligned} \quad \Longrightarrow \quad V_4 = \{ \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \mid b_1 = b_2 \text{ where } b_2 \in \mathbb{C} \}$$

Example If the matrix is upper diagonal or lower diagonal

$$T = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

then the polynomial is easy to factor.

$$0 = \begin{vmatrix} 2-x & 1 & 0 \\ 0 & 3-x & 1 \\ 0 & 0 & 2-x \end{vmatrix} = (3-x)(2-x)^2$$

These are the solutions for $\lambda_1 = 3$.

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Longrightarrow \quad V_3 = \{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} z_2 \mid z_2 \in \mathbb{C} \}$$

These are for $\lambda_2 = 2$.

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Longrightarrow \quad V_2 = \{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} z_1 \mid z_1 \in \mathbb{C} \}$$

Characteristic polynomial

3.9 *Definition* The *characteristic polynomial of a square matrix* T is the determinant $|T - xI|$ where x is a variable. The *characteristic equation* is $|T - xI| = 0$. The *characteristic polynomial of a transformation* t is the characteristic polynomial of any matrix representation $\text{Rep}_{B,B}(t)$.

Note Exercise 32 checks that the characteristic polynomial of a transformation is well-defined, that is, that the characteristic polynomial is the same no matter which basis we use for the representation.

3.10 *Lemma* A linear transformation on a nontrivial vector space has at least one eigenvalue.

Proof Any root of the characteristic polynomial is an eigenvalue. Over the complex numbers, any polynomial of degree one or greater has a root. QED

Remark This result is why we switched in this chapter from working with real number scalars to complex number scalars.

Eigenspace

3.12 *Definition* The *eigenspace of a transformation* t *associated with the eigenvalue* λ is $V_\lambda = \{\vec{\zeta} \mid t(\vec{\zeta}) = \lambda\vec{\zeta}\}$. The eigenspace of a matrix is analogous.

Example Recall that this matrix has three eigenvalues, 5, 4, and 2.

$$T = \begin{pmatrix} 0 & 5 & 7 \\ -2 & 7 & 7 \\ -1 & 1 & 4 \end{pmatrix}$$

Earlier, we found that these are the eigenspaces.

$$V_5 = \{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} c \mid c \in \mathbb{C}\} \quad V_4 = \{\begin{pmatrix} -7 \\ -7 \\ 1 \end{pmatrix} c \mid c \in \mathbb{C}\} \quad V_2 = \{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} c \mid c \in \mathbb{C}\}$$

3.13 *Lemma* An eigenspace is a subspace.

Proof Fix an eigenvalue λ. Notice first that V_λ contains the zero vector since $t(\vec{0}) = \vec{0}$, which equals $\lambda\vec{0}$. So the eigenspace is a nonempty subset of the space. What remains is to check closure of this set under linear combinations. Take $\vec{\zeta}_1, \ldots, \vec{\zeta}_n \in V_\lambda$ and then verify

$$\begin{aligned} t(c_1\vec{\zeta}_1 + c_2\vec{\zeta}_2 + \cdots + c_n\vec{\zeta}_n) &= c_1 t(\vec{\zeta}_1) + \cdots + c_n t(\vec{\zeta}_n) \\ &= c_1\lambda\vec{\zeta}_1 + \cdots + c_n\lambda\vec{\zeta}_n \\ &= \lambda(c_1\vec{\zeta}_1 + \cdots + c_n\vec{\zeta}_n) \end{aligned}$$

that the combination is also an element of V_λ. QED

3.17 *Theorem* For any set of distinct eigenvalues of a map or matrix, a set of associated eigenvectors, one per eigenvalue, is linearly independent.

Proof We will use induction on the number of eigenvalues. The base step is that there are zero eigenvalues. Then the set of associated vectors is empty and so is linearly independent.

Example This matrix from above has three eigenvalues, 5, 4, and 2.

$$T = \begin{pmatrix} 0 & 5 & 7 \\ -2 & 7 & 7 \\ -1 & 1 & 4 \end{pmatrix}$$

Picking a nonzero vector from each eigenspace we get this linearly independent set (which is a basis because it has three elements).

$$\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -14 \\ -14 \\ 2 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 1/2 \\ -1/2 \end{pmatrix} \}$$

Example This upper-triangular matrix has the eigenvalues 2 and 3

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

Picking a vector from each of V_3 and V_2 gives this linearly independent set.

$$\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \}$$