Inverses

Definition of matrix inverse

4.1 *Definition* A matrix G is a *left inverse matrix* of the matrix H if GH is the identity matrix. It is a *right inverse* if HG is the identity. A matrix H with a two-sided inverse is an *invertible matrix*. That two-sided inverse is denoted H^{-1}.

Example This matrix

$$H = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$$

has a two-sided inverse.

$$H^{-1} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$$

To check that, we multiply them in both orders. Here is one; the other is just as easy.

$$\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Example This matrix is nonsingular and so is invertible.

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 0 & -1 \\ 1 & 2 & 0 \end{pmatrix}$$

To ease the inverse calculation described in the prior proof, we write the matrix A next to the 3×3 identity and as we Gauss-Jordan reduce the matrix on the left, we apply those operations also on the right.

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 2 & 0 & -1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow[-\rho_1+\rho_3]{-2\rho_1+\rho_2} \left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -6 & -3 & -2 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{-1/6\rho_2+\rho_3} \left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -6 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1/2 & -2/3 & -1/6 & 1 \end{array}\right)$$

The right-hand side is in echelon form. We continue with the second half of Gauss-Jordan reduction on the next slide.

Example One advantage of knowing a matrix inverse is that it makes solving a linear system easy and quick. To solve

$$\begin{aligned} 2x + 5y &= -3 \\ x + 3y &= \ 10 \end{aligned}$$

rewrite as a matrix equation

$$\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 10 \end{pmatrix}$$

and multiply both sides (from the left) by the matrix inverse.

$$\begin{aligned} \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 10 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} -59 \\ 23 \end{pmatrix} \\ \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} -59 \\ 23 \end{pmatrix} \end{aligned}$$

$$\xrightarrow[-2\rho_3]{-1/6\rho_2} \left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1/2 & 1/3 & -1/6 & 0 \\ 0 & 0 & 1 & 4/3 & 1/3 & -2 \end{array}\right)$$

$$\xrightarrow[-(1/2)\rho_3+\rho_2]{-\rho_3+\rho_1} \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & -1/3 & -1/3 & 2 \\ 0 & 1 & 0 & -1/3 & -1/3 & 1 \\ 0 & 0 & 1 & 4/3 & 1/3 & -2 \end{array}\right)$$

$$\xrightarrow{-3\rho_2+\rho_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/3 & 2/3 & -1 \\ 0 & 1 & 0 & -1/3 & -1/3 & 1 \\ 0 & 0 & 1 & 4/3 & 1/3 & -2 \end{array}\right)$$

This is the inverse.

$$A^{-1} = \begin{pmatrix} 2/3 & 2/3 & -1 \\ -1/3 & -1/3 & 1 \\ 4/3 & 1/3 & -2 \end{pmatrix}$$

Example This is the same kind of calculation applied to a 2×2 matrix.

$$\left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 4 & 3 & 0 & 1 \end{array}\right) \xrightarrow{-2\rho_1+\rho_2} \left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right)$$

$$\xrightarrow{(1/2)\rho_1} \left(\begin{array}{cc|cc} 1 & 1/2 & 1/2 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right)$$

$$\xrightarrow{-(1/2)\rho_2+\rho_1} \left(\begin{array}{cc|cc} 1 & 0 & 3/2 & -1/2 \\ 0 & 1 & -2 & 1 \end{array}\right)$$

The inverse is this.

$$\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 3/2 & -1/2 \\ -2 & 1 \end{pmatrix}$$

Finding the inverse of a matrix A is a lot of work but as we noted earlier, once we have it then solving linear systems $A\vec{x} = \vec{b}$ is easy.

Example The linear system

$$\begin{aligned} x + 3y + z &= 2 \\ 2x \qquad - z &= 12 \\ x + 2y \qquad &= 4 \end{aligned}$$

is this matrix equation.

$$\begin{pmatrix} 1 & 3 & 1 \\ 2 & 0 & -1 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 12 \\ 4 \end{pmatrix}$$

Solve it by multiplying both sides from the left by the inverse that we found earlier.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2/3 & 2/3 & -1 \\ -1/3 & -1/3 & 1 \\ 4/3 & 1/3 & -2 \end{pmatrix} \begin{pmatrix} 2 \\ 12 \\ 4 \end{pmatrix} = \begin{pmatrix} 16/3 \\ -2/3 \\ -4/3 \end{pmatrix}$$

We sometimes want to repeatedly solve systems with the same left side but different right sides. This system equals the one on the prior slide but for one number on the right.

$$\begin{aligned} x+3y+z&=\ 1\\ 2x\qquad\quad -z&=12\\ x+2y\qquad\quad &=\ 4 \end{aligned}$$

The solution is this.

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2/3&2/3&-1\\-1/3&-1/3&1\\4/3&1/3&-2\end{pmatrix}\begin{pmatrix}1\\12\\4\end{pmatrix}=\begin{pmatrix}14/3\\-1/3\\-8/3\end{pmatrix}$$

The inverse of a 2×2 matrix

4.11 *Corollary* The inverse for a 2×2 matrix exists and equals

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

if and only if $ad - bc \neq 0$.

Proof This computation is Exercise 21 . QED

Example

$$\begin{pmatrix} 2 & 4 \\ -1 & 1 \end{pmatrix}^{-1} = \frac{1}{6}\begin{pmatrix} 1 & -4 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1/6 & -2/3 \\ 1/6 & 1/3 \end{pmatrix}$$

The 3×3 formula is much more complicated. We will cover it in the next chapter.

Four.I Definition of Determinant

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Warning

The formula for the determinant of a 2×2 matrix has something to do with multiplying diagonals.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Sometimes people have learned a mnemonic for the 3×3 formula that has to do with multplying diagonals.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - gec - hfa - idb \qquad \left(\begin{array}{ccc|cc} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{array}\right)$$

Don't try to extend to 4×4 or larger sizes. For those cases instead use Gauss's Method.