

Change of basis

①

Remember the matrix $A = \begin{pmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{pmatrix}$

Eigenvalue $\lambda_1 = 1$ basis for eigenspace $\left\{ \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \right\}$

$\lambda_2 = -3$ basis for eigenspace $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}$

Fact Putting together all of the bases for all of the eigenspaces gives a set that is still linearly independent:

$$B = \left\{ \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

This is 3 vectors that are linearly independent in \mathbb{R}^3

$\Rightarrow B$ is a basis for \mathbb{R}^3

If we can find a basis for everything made only of eigenvectors (i.e. if we have enough eigenvectors)

we say A is diagonalizable. Why?

Since B is a basis for \mathbb{R}^3 , for any \vec{v} in \mathbb{R}^3 there are a_1, a_2, a_3 with

$$\vec{v} = a_1 \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + a_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

We write $\text{Rep}_B(\vec{v}) = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$

Now remember that $A \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$ eigenvector with eigenvalue 1

$A \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = -3 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ eigenvectors with eigenvalue

$A \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = -3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$ -3

$$\begin{aligned} \text{So } A\vec{v} &= A \left(a_1 \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + a_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right) \\ &= a_1 A \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} + a_2 A \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + a_3 A \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \\ &= a_1 \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} - 3a_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} - 3a_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \end{aligned}$$

(2)

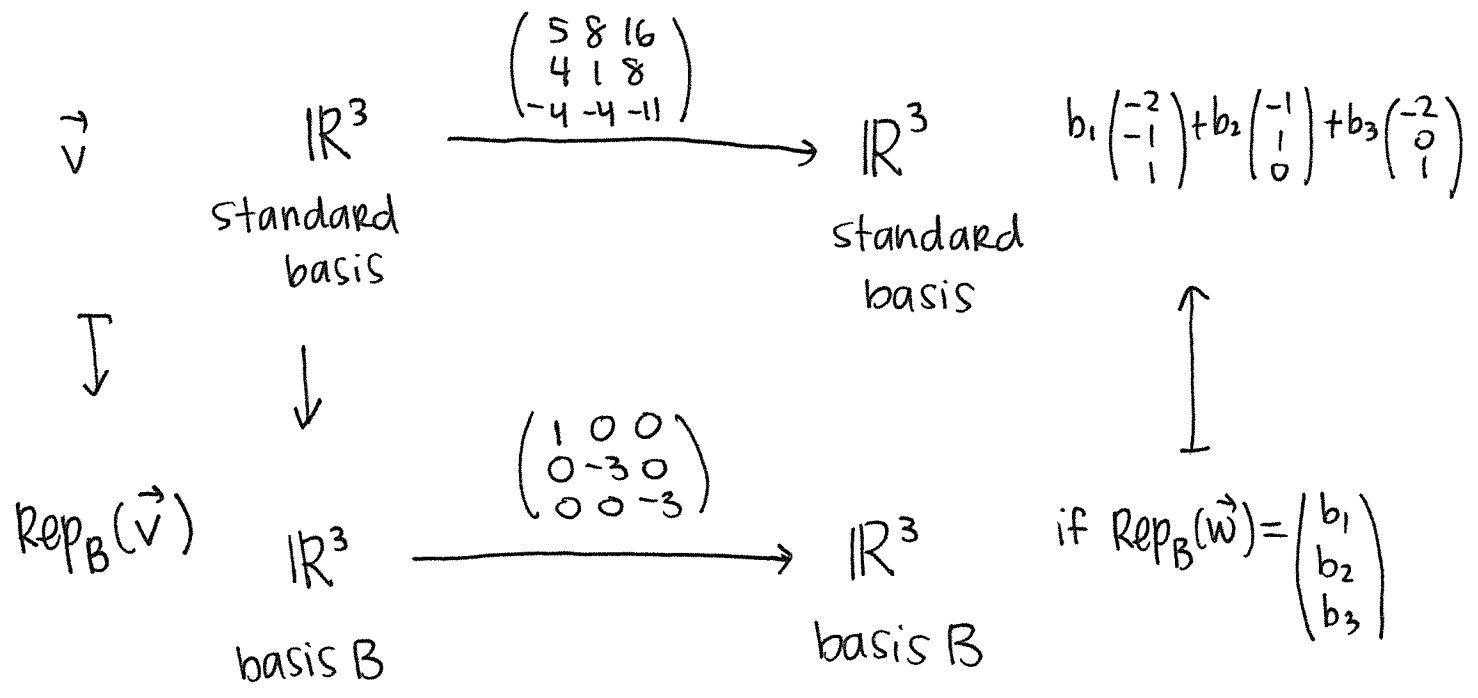
In other words

$$\text{Rep}_B(A\vec{v}) = \begin{pmatrix} a_1 \\ -3a_2 \\ -3a_3 \end{pmatrix}$$

In other other words, if we only worked with the basis B instead of the standard basis

A would be $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$ ← a diagonal matrix

A is diagonalizable because there is a basis on which it acts like a diagonal matrix!



(4)

Let's look at the vertical arrow on the left

If $\vec{v} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ in the standard basis

then $\text{Rep}_B \vec{v} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ where a_1, a_2, a_3 are the

solutions to $a_1 \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + a_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

i.e. if $C = \begin{pmatrix} -2 & -1 & -2 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ we solve $C\vec{a} = \vec{x}$
for \vec{a}

i.e. $\vec{a} = C^{-1}\vec{x}$

Key insight: The map $\vec{v} \mapsto \text{Rep}_B \vec{v}$

is given by the matrix C^{-1}

C^{-1} is a change-of-basis matrix. It changes from the standard basis to the basis B .

Now let's look at the vertical arrow on the right

It does

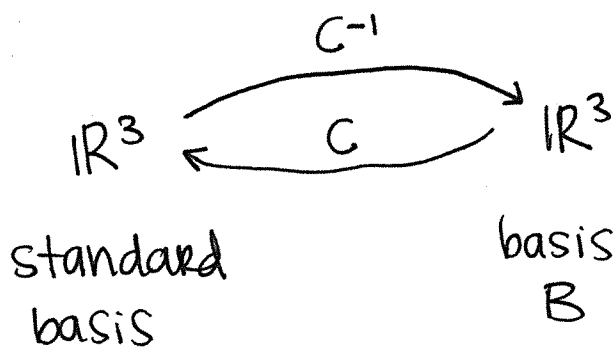
$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \mapsto b_1 \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} + b_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + b_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2b_1 - b_2 - 2b_3 \\ -b_1 + b_2 \\ b_1 + b_3 \end{pmatrix}$$

$$= C \vec{b}$$

Key insight: Going back from the basis B to the standard basis is given by the matrix $C = \begin{pmatrix} -2 & -1 & -2 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

It makes sense: C and C^{-1} undo each other



Every invertible matrix is a change-of-basis matrix

$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ goes from $B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \right\}$ to standard basis

A^{-1} goes from standard basis to basis B

Write $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$ $A = \begin{pmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & 11 \end{pmatrix}$ $C = \begin{pmatrix} -2 & -1 & -2 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ (6)

our diagram was

$$\begin{array}{ccc} \mathbb{R}^3 & \xrightarrow{A} & \mathbb{R}^3 \\ C^{-1} \downarrow & & \uparrow C \\ \mathbb{R}^3 & \xrightarrow{D} & \mathbb{R}^3 \end{array}$$

so $A = CDC^{-1}$ or $D = C^{-1}AC$

Since A & D are related by a change-of-basis, we say A and D are similar matrices

Remarks

- If A is not diagonalizable, A is not similar to any diagonal matrix, but it does have a Jordan Canonical Form (diagonal with maybe some 1's above the diagonal) That's almost as nice.
- Fun thing we can do:

$$\begin{aligned} A^{10} &= (CDC^{-1})^{10} = C \underbrace{DC^{-1}C}_{\text{cancel}} DC^{-1} \underbrace{C}_{\text{cancel}} DC^{-1} \dots C DC^{-1} \\ &= C D^{10} C^{-1} \end{aligned}$$

but $D^{10} = \begin{pmatrix} 1^{10} & 0 & 0 \\ 0 & (-3)^{10} & 0 \\ 0 & 0 & (-3)^{10} \end{pmatrix}$ ← easy to compute!

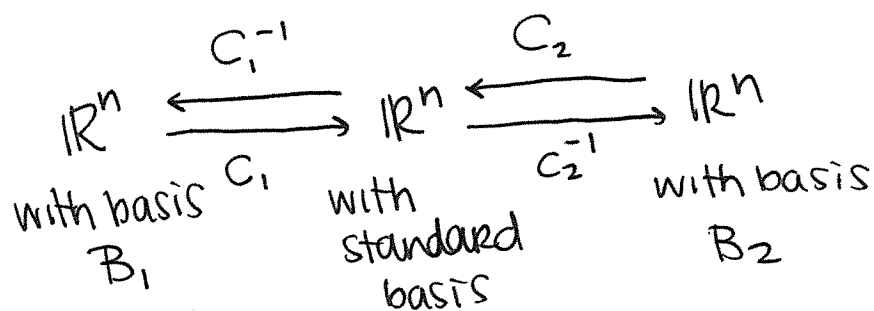
(7)

- What if you want to go between $\text{Rep}_{B_1}(\vec{v})$ and $\text{Rep}_{B_2}(\vec{v})$

where B_1 and B_2 are 2 bases but neither is the standard basis?

Let C_1 go between B_1 & standard basis

C_2 go between B_2 & standard basis



Then $C_2^{-1}C_1$ is the matrix that converts from B_1 to B_2

and $C_1^{-1}C_2$ is the matrix that converts from B_2 to B_1