Solving a linear system

Example To find the solution of this system

$$\begin{aligned} (1/4)x + y - z &= 0 \\ x + 4y + 2z &= 12 \\ 2x - 3y - z &= 3 \end{aligned}$$

we transform it to one whose solution is easy. Start by clearing the fraction.

$$\xrightarrow{4\rho_1} \quad \begin{aligned} x + 4y - 4z &= 0 \\ x + 4y + 2z &= 12 \\ 2x - 3y - z &= 3 \end{aligned}$$

Next use the first row to act on the rows below, eliminating their x terms.

$$\xrightarrow[-2\rho_1+\rho_3]{-1\rho_1+\rho_2} \quad \begin{aligned} x + \quad 4y - 4z &= 0 \\ + 6z &= 12 \\ -11y + 7z &= 3 \end{aligned}$$

Then swap to bring a y term to the second row.

$$\xrightarrow{\rho_2\leftrightarrow\rho_3}\quad \begin{aligned} x + \quad 4y - 4z &= 0 \\ -11y + 7z &= 3 \\ 6z &= 12 \end{aligned}$$

Now solve the bottom row: $z = 2$. With that, the shape of the transformed system lets us solve for y by substituting into the second row: $-11y + 7(2) = 3$ shows $y = 1$. The shape also lets us solve for x by substituting into the first row: $x + 4(1) - 4(2) = 0$, so that $x = 4$.

1.10 *Definition* In each row of a system, the first variable with a nonzero coefficient is the row's *leading variable*. A system is in *echelon form* if each leading variable is to the right of the leading variable in the row above it, except for the leading variable in the first row, and any all-zero rows are at the bottom.

Gauss's Method

1.5 *Theorem* If a linear system is changed to another by one of these operations

1) an equation is swapped with another
2) an equation has both sides multiplied by a nonzero constant
3) an equation is replaced by the sum of itself and a multiple of another

then the two systems have the same set of solutions.

See the book for the proof.

1.6 *Definition* The three operations from Theorem 1.5 are the *elementary reduction operations*, or *row operations*, or *Gaussian operations*. They are *swapping*, *multiplying by a scalar* (or *rescaling*), and *row combination*.

Example

$$\begin{array}{rcl} 2x-3y-z+2w&=&-2\\ x+3z+1w&=&6\\ 2x-3y-z+3w&=&-3\\ y+z-2w&=&4 \end{array} \quad \xrightarrow[-\rho_1+\rho_3]{(-1/2)\rho_1+\rho_2} \quad \begin{array}{rcl} 2x-3y-z+2w&=&-2\\ (3/2)y+(7/2)z&=&7\\ w&=&-1\\ y+z-2w&=&4 \end{array}$$

$$\xrightarrow{(-2/3)\rho_2+\rho_4} \quad \begin{array}{rcl} 2x-3y-z+2w&=&-2\\ (3/2)y+(7/2)z&=&7\\ w&=&-1\\ -(4/3)z-2w&=&-2/3 \end{array}$$

$$\xrightarrow{\rho_3\leftrightarrow\rho_4} \quad \begin{array}{rcl} 2x-3y-z+2w&=&-2\\ (3/2)y+(7/2)z&=&7\\ -(4/3)z-2w&=&-2/3\\ w&=&-1 \end{array}$$

The fourth equation says $w=-1$. Substituting back into the third equation gives $z=2$. Then back substitution into the second and first rows gives $y=0$ and $x=1$. The unique solution is $(1,0,2,-1)$.

Systems without a unique solution

Example This system has no solution.

$$\begin{aligned} x + y + z &= 6 \\ x + 2y + z &= 8 \\ 2x + 3y + 2z &= 13 \end{aligned}$$

On the left side of the equals sign the sum of the first two rows equals the third row, while on the right that is not so. So there is no triple of reals that makes all three equations true.

Gauss' Method makes the inconsistency clear.

$$\xrightarrow[-2\rho_1+\rho_3]{-\rho_1+\rho_2} \begin{array}{rcl} x + y + z &=& 6 \\ y &=& 2 \\ y &=& 1 \end{array} \xrightarrow{-\rho_2+\rho_3} \begin{array}{rcl} x + y + z &=& 6 \\ y &=& 2 \\ 0 &=& -1 \end{array}$$

Example This system has infinitely many solutions.

$$\begin{array}{rcl} x - y + z = 4 \\ x + y - 2z = -1 \end{array} \xrightarrow{-\rho_1+\rho_2} \begin{array}{rcl} x - y + z = 4 \\ 2y - 3z = -5 \end{array}$$

Taking $z = 0$ gives the solution $(3/2, -5/2, 0)$. Taking $z = -1$ gives $(1, -4, -1)$.

Example Another system with infinitely many solutions.

$$\begin{array}{r} -x - y + 3z = 3 \\ x + z = 3 \\ 3x - y + 7z = 15 \end{array} \xrightarrow[3\rho_1+\rho_3]{\rho_1+\rho_2} \begin{array}{r} -x - y + 3z = 3 \\ -y + 4z = 6 \\ -4y + 16z = 24 \end{array}$$

$$\xrightarrow{-4\rho_2+\rho_3} \begin{array}{r} -x - y + 3z = 3 \\ -y + 4z = 6 \\ 0 = 0 \end{array}$$

Taking $z = 0$ gives $(3, -6, 0)$ while taking $z = 1$ gives $(2, -2, 1)$.