MATH 320
Section 7.3 #26.

We are trying to solve

$$\underline{x}'(t) = \begin{bmatrix} 3 & 0 & 1 \\ 9 & -1 & 2 \\ -9 & 4 & -1 \end{bmatrix} \underline{x}(t), \quad \underline{x}(0) = \begin{bmatrix} 0 \\ 0 \\ 17 \end{bmatrix}.$$

We first find the eigenvalues of the matrix:

$$\begin{vmatrix} 3-\lambda & 0 & 1 \\ 9 & -1-\lambda & 2 \\ -9 & 4 & -1-\lambda \end{vmatrix} = (3-\lambda)\left[(-1-\lambda)^2 - 8\right] + \left[36 - (-9)(-1-\lambda)\right]$$

$$= (3-\lambda)(\lambda^2 + 2\lambda - 7) + 9(3-\lambda)$$
$$= (3-\lambda)(\lambda^2 + 2\lambda + 2)$$
$$= (3-\lambda)\underbrace{(\lambda+1+i)(\lambda+1-i)}_{\text{using quadratic formula.}}$$

So we have $\lambda_1 = 3, \quad \lambda_2 = -1-i, \quad \lambda_3 = -1+i.$

We now find eigenvectors:

$\lambda_1 = 3$: Solve $\begin{bmatrix} 0 & 0 & 1 \\ 9 & -4 & 2 \\ -9 & 4 & -4 \end{bmatrix} \underline{\xi} = 0$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 9 & -4 & 2 & 0 \\ -9 & 4 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 9 & -4 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ -9 & 4 & -4 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 9 & -4 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 9 & -4 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so if $\underline{\xi} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, $c = 0$, $b = t$,

$a = \frac{1}{9}(4t - 2(0))$

$a = \frac{4}{9}t$

The eigenspace is given by $\begin{bmatrix} 4/9 \\ 1 \\ 0 \end{bmatrix} t$

Choosing $t = 9$, we get the eigenvector

$\begin{bmatrix} 4 \\ 9 \\ 0 \end{bmatrix}$ associated to $\lambda = 3$

$\lambda_2 = -1-i$, $\lambda_3 = -1+i$ is a complex conjugate pair, so we pick one of the two, find the solution associated to it and then the real and imaginary parts give us 2 linearly independent real solutions.

So I pick $\lambda_3 = -1+i$. I solve

$$\begin{bmatrix} 4-i & 0 & 1 \\ 9 & -i & 2 \\ -9 & 4 & -i \end{bmatrix} \underline{\xi} = 0$$

$$\left[\begin{array}{ccc|c} 4-i & 0 & 1 & 0 \\ 9 & -i & 2 & 0 \\ -9 & 4 & -i & 0 \end{array}\right] \underset{R_3+R_1}{\overset{R_1-\frac{4}{9}R_2}{\sim}} \left[\begin{array}{ccc|c} -i & 4i/9 & 1/9 & 0 \\ 9 & -i & 2 & 0 \\ 0 & 4-i & 2-i & 0 \end{array}\right]$$

$$\overset{9iR_1}{\sim} \left[\begin{array}{ccc|c} 9 & -4 & i & 0 \\ 9 & -i & 2 & 0 \\ 0 & 4-i & 2-i & 0 \end{array}\right]$$

$$\overset{R_2-R_1}{\sim} \left[\begin{array}{ccc|c} 9 & -4 & i & 0 \\ 0 & 4-i & 2-i & 0 \\ 0 & 4-i & 2-i & 0 \end{array}\right]$$

$$\overset{R_3-R_2}{\sim} \left[\begin{array}{ccc|c} 9 & -4 & i & 0 \\ 0 & 4-i & 2-i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so if $\underline{\xi} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, $c = t$

$$b = \frac{-2+i}{4-i}\, t$$

$$a = \frac{1}{9}\left[4\left(\frac{-2+i}{4-i}\right)t - it\right]$$

I need to simplify these numbers, mostly by getting rid of the $4-i$ in the denominator. I do this by multiplying top and bottom by the conjugate of $4-i$:

$$\frac{-2+i}{4-i} \cdot \frac{4+i}{4+i} = \frac{-8-2i+4i-1}{16+4i-4i+1}$$
$$= \frac{-9+2i}{17}$$

so $a = \frac{1}{9}\left(4\left(\frac{-9+2i}{17}\right)t - it\right)$

$$= \left(\frac{-36+8i}{153}\right)t - \frac{17i\,t}{153} = \left(\frac{-36-9i}{153}\right)t$$
$$= \frac{-4-i}{17}\, t$$

$$b = \frac{-9+2i}{17}\, t$$

So the eigenspace is given by

$$\begin{bmatrix} \frac{-4-i}{17} \\ \frac{-9+2i}{17} \\ 1 \end{bmatrix} t$$

I choose $t=17$ to get the eigenvector

$$\begin{bmatrix} -4-i \\ -9+2i \\ 17 \end{bmatrix}$$ associated to $\lambda_3 = -1+i$.

So the complex solution associated to this eigenvalue is

$$e^{(-1+i)t} \begin{bmatrix} -4-i \\ -9+2i \\ 17 \end{bmatrix}$$

I now find its real and complex parts to get 2 linearly independent real solutions:

$$e^{(-1+i)t} \begin{bmatrix} -4-i \\ -9+2i \\ 17 \end{bmatrix} = \begin{bmatrix} e^{-t}(\cos t + i\sin t)(-4-i) \\ e^{-t}(\cos t + i\sin t)(-9+2i) \\ e^{-t}(\cos t + i\sin t)(17) \end{bmatrix}$$

$$= \begin{bmatrix} -4e^{-t}\cos t - ie^{-t}\cos t - 4ie^{-t}\sin t + e^{-t}\sin t \\ -9e^{-t}\cos t + 2ie^{-t}\cos t - 9ie^{-t}\sin t - 2e^{-t}\sin t \\ 17e^{-t}\cos t \qquad + 17ie^{-t}\sin t \end{bmatrix}$$

$$= \underbrace{e^{-t}\begin{bmatrix} -4\cos t + \sin t \\ -9\cos t - 2\sin t \\ 17\cos t \end{bmatrix}}_{\text{real part}} + \underbrace{ie^{-t}\begin{bmatrix} -4\sin t - \cos t \\ -9\sin t + 2\cos t \\ 17\sin t \end{bmatrix}}_{\text{imaginary part.}}$$

I don't do anything for $\lambda_2 = -1-i$ since I would get the same real and imaginary parts. Besides, I already have 3 linearly independent solutions which is all I need.

So the general solution is

$$\underline{x}(t) = c_1 e^{3t}\begin{bmatrix} 4 \\ 9 \\ 0 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} -4\cos t + \sin t \\ -9\cos t - 2\sin t \\ 17\cos t \end{bmatrix}$$

$$+ c_3 e^{-t}\begin{bmatrix} -4\sin t - \cos t \\ -9\sin t + 2\cos t \\ 17\sin t \end{bmatrix}$$

We now plug in $t=0$ to find c_1, c_2, c_3.

$$\begin{bmatrix} 0 \\ 0 \\ 17 \end{bmatrix} = c_1 \begin{bmatrix} 4 \\ 9 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -4 \\ -9 \\ 17 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

So we are solving

$$\begin{bmatrix} 4 & -4 & -1 \\ 9 & -9 & 2 \\ 0 & 17 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 17 \end{bmatrix}$$

Using Gaussian elimination:

$$\begin{bmatrix} 4 & -4 & -1 & 0 \\ 9 & -9 & 2 & 0 \\ 0 & 17 & 0 & 17 \end{bmatrix} \sim \begin{bmatrix} 4 & -4 & -1 & 0 \\ 0 & 0 & 17/4 & 0 \\ 0 & 17 & 0 & 17 \end{bmatrix}$$

So $c_3 = 0$, $c_2 = 1$, $c_1 = \frac{1}{4}(4\cdot(1) + 1(0)) = 1$

and the solution is

$$\underline{x}(t) = e^{3t} \begin{bmatrix} 4 \\ 9 \\ 0 \end{bmatrix} + e^{-t} \begin{bmatrix} -4\cos t + \sin t \\ -9\cos t - 2\sin t \\ 17\cos t \end{bmatrix}$$