NAME:

To get the most of this quiz, allow yourself no more than 20 minutes to completely answer it, and do not use any notes or outside help. I will grade it if you hand it in during discussion on May 4 or 6.

Problem 1 (10 points): *Consider the system of differential equations*

$$\begin{aligned} x_1' &= x_1 + 2x_2 \\ x_2' &= 2x_1 + x_2 \end{aligned}$$

a) *Write this as* $\boldsymbol{x}'(t)=\boldsymbol{P}(t)\boldsymbol{x}(t)$.

b) *Your matrix* $\boldsymbol{P}$ *should in fact be constant (all of its entries should be constants, not functions of* t*). Find its eigenvalues and a basis for its eigenspaces.*

c) *Check that*

$$\boldsymbol{x}_1(t) = e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{x}_2(t) = e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

are two linearly independent solutions of this system. Armed with this knowledge, write the general solution of this system.

Solution:

a) We have

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

b) We first find the characteristic polynomial:

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 = (\lambda-3)(\lambda+1)$$

So the two eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -1$.
The eigenspace associated to the first eigenvalue is found by finding all vectors $\mathbf{v}= (v_1, v_2)$ such that:

$$\begin{bmatrix} 1-3 & 2 \\ 2 & 1-3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

To do this we row-reduce

$$\begin{bmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} -2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So v_2 is a free variable, say $v_2 = s$, and back-solving for v_1 we get that $v_1 = s$ as well. The eigenspace is given by all vectors that can be written as

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} s$$

and a basis for the eigenspace is the vector $(1, 1)$.
The eigenspace associated to the second eigenvalue is found by finding all vectors $\mathbf{v}= (v_1, v_2)$ such that:

$$\begin{bmatrix} 1+1 & 2 \\ 2 & 1+1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

To do this we row-reduce

$$\begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So v_2 is a free variable, say $v_2 = s$, and back-solving for v_1 we get that $v_1 = -s$. The eigenspace is given by all vectors that can be written as

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} s$$

and a basis for the eigenspace is the vector $(-1, 1)$.

c) We must first check that $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are solutions to the equation.
We have that on the one hand

$$\mathbf{x}_1'(t) = \begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix}$$

and on the other hand,

$$\mathbf{P}\mathbf{x}_1(t) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix} = \begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix}$$

Since these are equal $\mathbf{x}_1(t)$ is a solution of the system.
For $\mathbf{x}_2(t)$, we have

$$\mathbf{x}_2'(t) = \begin{bmatrix} 3e^{3t} \\ 3e^{3t} \end{bmatrix}$$

and

$$\mathbf{P}\mathbf{x}_2(t) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} = \begin{bmatrix} 3e^{3t} \\ 3e^{3t} \end{bmatrix}$$

Since these are equal $\mathbf{x}_2(t)$ is a solution of the system.
We now check that the two given solutions are linearly independent by computing the Wronskian

$$\begin{vmatrix} e^{-t} & e^{3t} \\ -e^{-t} & e^{3t} \end{vmatrix} = e^{2t} + e^{2t} = 2e^{2t}$$

Since this is not zero, the solutions are linearly independent.
Because we have two linearly independent solutions to a system of two first-order differentials equations, we know that they span the whole solution space, and all solutions can be written as:

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1 \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix}$$