MATH 320 QUIZ 1 DUE DATE: FEBRUARY 2 OR 4

NAME:

To get the most of this quiz, allow yourself no more than 20 minutes to completely answer it, and do not use any notes or outside help. I will grade it if you hand it in during discussion on February 4.

Problem 1 (10 points): *Find the solution to the differential equation*

$$(1+x)^2\frac{dy}{dx} = (1+y)^2, \quad y(0) = -\frac{1}{2}.$$

Give the region of validity of the solution, and use Theorem 1 from Section 1.3 to argue that this is the only solution of this equation in this region.

Solution: This DE is separable, so we may solve it using the techniques of Section 1.4. We integrate:

$$\int \frac{1}{(1+y)^2}\,dy = -\frac{1}{1+y} + C$$

so that a general solution to the DE without the initial value is

$$\frac{1}{1+y} = \frac{1}{1+x} + C. \tag{1}$$

We now plug in the initial value to get that $C = 1$.
We must now investigate the region of validity of the solution. From looking at equation (1), it seems that the solution would not be defined at $(-1,-1)$. If we rewrite the solution as

$$y = -\frac{1}{x+2} \tag{2}$$

though, we see that $(-1,-1)$ is not really a problem; rather the solution is not defined at $x = -2$. (We could have seen this from equation (1) by noticing that it makes the right-hand side zero, while the left-hand side can never be zero. Also, $(-1,-1)$ is not really a problem because it is a problem we have created ourselves when we separated variables. Plugging in $(-1,-1)$ into the original equation gives $0 = 0$, which is fine.) Since our initial value is at $x = 0$, we need to pick the right side of our problem point, so that the solution is valid on $x > -2$.
To apply Theorem 1, we rewrite the DE as

$$\frac{dy}{dx} = \left(\frac{1+y}{1+x}\right)^2$$

So here $f(x,y) = \left(\frac{1+y}{1+x}\right)^2$, which is continuous if $x \neq -1$, and $\frac{\partial f}{\partial y} = \frac{2(1+y)}{(1+x)^2}$, which is continuous if $x \neq -1$. Hence on the rectangle $-1 < x < \infty$, $-\infty < y < \infty$, the solution, where it exists, is unique. (Note that Theorem 1 from Section 1.3 only guarantees existence in a subinterval containing the initial value.) A solution exists in these intervals since we have found one, and so our solution is unique where $-1 < x < \infty$. The theorem does not allow us to say anything for values of x between -2 and -1.