DATE: 4/21/2009 Linear Algebra MATH 320

This examines the connections between a few of the concepts we have learned in linear algebra this semester. Being very comfortable with these questions, their answers, and the ideas behind them will go a long way towards helping you do well on linear algebra questions in this class and more generally will give you a strong basis in linear algebra whenever it comes up again in your life. I will not post answers to these questions (so please don't ask me to); if you would like to know how you are doing or if you would like to discuss the answers to these questions, Professor Smith, Sarah and I will be more than happy to meet with you.

Throughout, let $\mathbf{A}$ be a square $n \times n$ matrix. Here are ten properties that are equivalent to $\mathbf{A}$ being invertible. (What I mean by this is that if $\mathbf{A}$ has any of the ten properties below, then it has the other nine properties and it is invertible, and if $\mathbf{A}$ is invertible, then it has all the ten properties below. In other words, being invertible and these ten properties are all the same thing.)

1) $\mathbf{A}$ is row-equivalent to the $n \times n$ identity matrix.
2) $\mathbf{A}$ has n leading entries.
3) $\det \mathbf{A} \neq 0$.
4) The rank of $\mathbf{A}$ is n.
5) The equation $\mathbf{Ax}$=0 only has the trivial solution $\mathbf{x}$=0.
6) The equation $\mathbf{Ax}$=$\mathbf{b}$ has exactly one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
7) The columns of $\mathbf{A}$ are linearly independent.
8) The columns of $\mathbf{A}$ span $\mathbb{R}^n$.
9) The columns of $\mathbf{A}$ form a basis of $\mathbb{R}^n$.
10) The number 0 is not an eigenvalue of $\mathbf{A}$.

Determinant and Inverses

If $\det \mathbf{A} = 10$, is $\mathbf{A}$ invertible?
If $\det \mathbf{A} = \pi$, is $\mathbf{A}$ invertible?
If $\det \mathbf{A} = 0$, is $\mathbf{A}$ invertible?

Row operations and determinant

How many different kinds of row operations are there and how does each change the determinant? So overall, what are the ways in which row operations can change the determinant?

Can a matrix with a non-zero determinant be row equivalent to a matrix with zero determinant? Can a matrix with zero determinant be row equivalent to a matrix with non-zero determinant?

Are all matrices with non-zero determinant row equivalent? If **A** is row equivalent to **B** and **B** is row equivalent to **C**, is **A** row equivalent to **C**?

Are all matrices with determinant zero row equivalent? (They aren't. Give an example of two 3×3 matrices that both have determinant zero and that are *not* row equivalent.)

Column space and row space

If **A** is row equivalent to **B**, do **A** and **B** have the same row space? Do **A** and **B** have the same column space? (You can find the answer to these questions in the book.)

If **A** is row equivalent to **B**, do **A** and **B** have the same rank?

What is the rank of the $n \times n$ identity matrix? If **A** is row equivalent to the identity matrix, what is the rank of **A**?

If **A** has k leading entries, what is the rank of **A**?

Linear independence, spanning and bases

In $\mathbb{R}^3$, can 4 vectors be linearly independent? Can 3 vectors be linearly independent? Can 2 vectors be linearly independent? Can 3 vectors be linearly dependent?
More generally, in $\mathbb{R}^n$, how many vectors can be linearly independent, and at which point do you know for sure the vectors are linearly dependent?

Can 4 vectors span all of $\mathbb{R}^3$? Can 3 vectors span all of $\mathbb{R}^3$? Can 2 vectors span all of $\mathbb{R}^3$? Is it possible for 3 vectors to not span all of $\mathbb{R}^3$?
More generally, in $\mathbb{R}^n$, how many vectors are necessary to span all of $\mathbb{R}^n$?

If I have 3 linearly independent vectors in $\mathbb{R}^3$, do they span $\mathbb{R}^3$? What can you say in the case of $\mathbb{R}^n$?

If 3 vectors span $\mathbb{R}^3$, are they linearly independent? Again, what can you say in the case of $\mathbb{R}^n$?

Write down the definition of a basis.

If I have 3 linearly independent vectors in $\mathbb{R}^3$, do they form a basis of $\mathbb{R}^3$? What if I have 2 linearly independent vectors?

If 3 vectors span $\mathbb{R}^3$, do they form a basis of $\mathbb{R}^3$? What if 4 vectors span $\mathbb{R}^3$, do they form a basis then?

Look at facts 7), 8) and 9). Does it make sense that they are all the same now?

Solutions and column spaces

Let $\mathbf{A}$ be any $n \times n$ matrix. How many solutions can the system $\mathbf{Ax}$=$\mathbf{b}$ have, if $\mathbf{b}$ is any vector? How many solutions can the system $\mathbf{Ax}$=0 have?

If $\mathbf{Ax}$=0 has infinitely many solutions, what is the determinant of $\mathbf{A}$? If $\mathbf{Ax}$=0 has infinitely many solutions, is $\mathbf{A}$ row equivalent to the identity matrix?

Suppose I am row-reducing the matrix $\mathbf{A}$, and I end up with n leading entries (remember $\mathbf{A}$ is an $n \times n$ matrix). Is $\mathbf{A}$ invertible? What is the column space of $\mathbf{A}$? What are two different bases you could give for the column space of $\mathbf{A}$? How many solutions does the system $\mathbf{Ax}$=0 have?

Suppose that I am row-reducing the matrix $\mathbf{A}$, and I end up with 2 rows of zeros when I am done. How many leading entries does $\mathbf{A}$ have? How many solutions does $\mathbf{Ax}$=0 have? The vectors $\mathbf{x}$ such that $\mathbf{Ax}$=0 form a subspace of $\mathbb{R}^n$ (prove this); what is its dimension? What is the dimension of the column space of $\mathbf{A}$? If you had to give a basis for the column space, how many vectors would be in it?
(For this set of questions, you might want to work out a few examples of matrices with two rows of zeros when they are row reduced and see what happens.)

More generally, suppose that I am row-reducing the matrix $\mathbf{A}$ and I end up with k rows of zeros when I am done, where $0 \leq k \leq n$. How many leading entries does $\mathbf{A}$ have? What is the dimension of the subspace of $\mathbb{R}^n$ formed by the vectors $\mathbf{x}$ such that $\mathbf{Ax}$=0? What is the dimension of the column space of $\mathbf{A}$?

Eigenvalues

What does it mean if 0 is an eigenvalue of $\mathbf{A}$? What does it have to do with Fact 5)?