DATE: 2/16/2009 — The Euler Schemes — MATH 320

We will use the various Euler schemes we have learned about in class to estimate the value of y at $x = 2$ for the following DE:

$$y\,y' = 2x^3 \quad y(1) = 3$$

To evaluate how good the different Euler schemes are, we will first find the exact solution of this DE. Notice that "in real life" you will use the Euler method when you cannot find an exact solution for the DE. To solve this DE, you must first notice that it is a separable equation. From there solving the equation should be pretty straightforward but if at this point you are uncertain of how to proceed you should definitely talk to your TA or a friend in the class.

The exact solution to this DE is

$$y = \sqrt{x^4 + 8}$$

So that $y(2) = 4.898979...$

Note that for all three schemes we will have $x_0 = 1$, $y_0 = 3$, $h = 0.01$ so that we will need 100 steps to get an approximation for $y(2)$, and $f(x, y) = \frac{2x^3}{y}$.

The first sheet presents the Forward Euler scheme. In this scheme we have:

$$\begin{aligned} \tilde{y}_{n+1} &= \tilde{y}_n + h\,f(x_n, \tilde{y}_n) \\ &= \tilde{y}_n + 0.01\,\frac{2x_n^3}{\tilde{y}_n} \end{aligned}$$

To implement the scheme in Excel, all we need to do is fill out the first two lines (Steps 0 and 1), then select the 2 cells under Steps and the 2 cells under X and drag down the tiny square in the bottom right corner until you have 100 steps. Finally, drag down the tiny square in the bottom right corner of the "Step 1 of Y" cell. (You should try it, it's all set up in rows E, F and G.) Notice that here the percentage error is about 0.2%.

The second sheet presents the Backward Euler scheme. In this scheme we have:

$$\begin{aligned} \tilde{y}_{n+1} &= \tilde{y}_n + h\,f(x_{n+1}, \tilde{y}_{n+1}) \\ &= \tilde{y}_n + 0.01\,\frac{2x_{n+1}^3}{\tilde{y}_{n+1}} \end{aligned}$$

Notice that since this scheme is *implicit* there are $\tilde{y}_{n+1}$'s on both sides of the equation, which means that to get $\tilde{y}_{n+1}$, we must solve for $\tilde{y}_{n+1}$. We do this by multiplying through by $\tilde{y}_{n+1}$, and bringing everything to the same side. This gives us the equation:

$$(\tilde{y}_{n+1})^2 - \tilde{y}_n\,\tilde{y}_{n+1} - 0.02\,x_{n+1}^3 = 0$$

Since we know $\tilde{y}_n$ and x_{n+1}^3, this is simply a quadratic equation in our unknown, $\tilde{y}_{n+1}$. We can solve this equation using the quadratic formula, being careful to pick the positive square root in this case. (We pick the positive square root since the slope is positive, so we expect $\tilde{y}_{n+1}$ to be slightly bigger than $\tilde{y}_n/2$, not smaller.) So we get:

$$\tilde{y}_{n+1} = \frac{\tilde{y}_n + \sqrt{(\tilde{y}_n)^2 + 0.08\, x_{n+1}^3}}{2}$$

We can implement this scheme in Excel in the same way we implemented the Forward Euler scheme. We get a percentage error of about 0.2%, which is comparable to the Forward Euler scheme, as we would expect from the theory.

Finally the last sheet presents the Improved Euler scheme. In this last scheme we have:

$$\begin{aligned}
\tilde{y}_{n+1} &= \tilde{y}_n + \frac{h}{2}\left(f(x_n, \tilde{y}_n) + f(x_{n+1}, \tilde{u}_{n+1})\right) \\
&= \tilde{y}_n + 0.005\left(\frac{2x_n^3}{\tilde{y}_n} + \frac{2x_{n+1}^3}{\tilde{u}_{n+1}}\right) \\
&= \tilde{y}_n + 0.005\left(\frac{2x_n^3}{\tilde{y}_n} + \frac{2x_{n+1}^3}{\tilde{y}_n + 0.01\,\frac{2x_n^3}{\tilde{y}_n}}\right) \\
&= \tilde{y}_n + 0.005\left(\frac{2x_n^3}{\tilde{y}_n} + \frac{2x_{n+1}^3\,\tilde{y}_n}{\tilde{y}_n^2 + 0.02\,x_n^3}\right)
\end{aligned}$$

since

$$\tilde{u}_{n+1} = \tilde{y}_n + h\,f(x_n, \tilde{y}_n) = \tilde{y}_n + 0.01\,\frac{2x_n^3}{\tilde{y}_n}$$

In this last case we see that the percentage error is 0.0006% which is way better for not that much more work.