MATH 320
Section 4.5
Supplementary Problems

#1. $\underline{\underline{A}} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & -9 \\ 2 & 5 & 2 \end{bmatrix}$

The row space of $\underline{\underline{A}}$ is the subspace of $\mathbb{R}^3$ spanned by the vectors

$$\underline{v_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \underline{v_2} = \begin{bmatrix} 1 \\ 5 \\ -9 \end{bmatrix} \quad \underline{v_3} = \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}$$

that is, all vectors that can be written as

$$a\underline{v_1} + b\underline{v_2} + c\underline{v_3}.$$

Thus if $\underline{w}$ is in the row space of $\underline{\underline{A}}$, there are numbers a, b, c such that

$$\underline{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = a\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + b\begin{bmatrix} 1 \\ 5 \\ -9 \end{bmatrix} + c\begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}$$

or

$$\begin{aligned} a + b + 2c &= w_1 \\ 2a + 5b + 5c &= w_2 \\ 3a - 9b + 2c &= w_3 \end{aligned}$$

In matrix form, this is

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 5 & 5 \\ 3 & -9 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

So we row-reduce

$$\left[\begin{array}{cccc} 1 & 1 & 2 & w_1 \\ 2 & 5 & 5 & w_2 \\ 3 & -9 & 2 & w_3 \end{array}\right] \sim \left[\begin{array}{cccc} 1 & 1 & 2 & w_1 \\ 0 & 3 & 1 & w_2-2w_1 \\ 0 & -12 & -4 & w_3-3w_1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc} 1 & 1 & 2 & w_1 \\ 0 & 3 & 1 & w_2-2w_1 \\ 0 & 0 & 0 & (w_3-3w_1)+4(w_2-2w_1) \end{array}\right]$$

Because the last row is all zeroes, this means that if $\underline{w}$ is in the space, it must have:

$$w_3+4w_2-11w_1=0.$$

These are the $\underline{w}$'s I am interested in. In other words, the row space of $\underline{\underline{A}}$ is all vectors $\underline{w}$ such that $w_3+4w_2-11w_1=0$.

If $w_2=s$ and $w_3=t$, then $w_1=\frac{1}{11}(4s+t)$

So we row-reduce:

$$\begin{bmatrix} 1 & 2 & 3 & w_1 \\ 1 & 5 & -9 & w_2 \\ 2 & 5 & 2 & w_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & w_1 \\ 0 & 3 & -12 & w_2 - w_1 \\ 0 & 1 & -4 & w_3 - 2w_1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & w_1 \\ 0 & 1 & -4 & w_3 - 2w_1 \\ 0 & 3 & -12 & w_2 - w_1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & w_1 \\ 0 & 1 & -4 & w_3 - 2w_1 \\ 0 & 0 & 0 & (w_2 - w_1) - 3(w_3 - 2w_1) \end{bmatrix}$$

Again since the last row is all zeroes, for $\underline{w}$ to be in the column space it must have

$$w_2 + 5w_1 - 3w_3 = 0$$

So the column space of $\underline{\underline{A}}$ is all vectors $\underline{w}$ such that $5w_1 + w_2 - 3w_3 = 0$.

If $w_2 = s$ and $w_3 = t$, then $w_1 = \frac{1}{5}(3t - s)$

and $$\underline{w} = \begin{bmatrix} 3/5 \\ 0 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1/5 \\ 1 \\ 0 \end{bmatrix} s$$

So $\underline{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1/11 \\ 0 \\ 1 \end{bmatrix} t + \begin{bmatrix} 4/11 \\ 1 \\ 0 \end{bmatrix} s$

and $\left\{ \begin{bmatrix} 1/11 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4/11 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for the row space of $\underline{\underline{A}}$.

The column space of $\underline{\underline{A}}$ is the subspace of $\mathbb{R}^3$ spanned by the vectors

$$\underline{u_1} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \underline{u_2} = \begin{bmatrix} 3 \\ 5 \\ 5 \end{bmatrix} \quad \underline{u_3} = \begin{bmatrix} 3 \\ -9 \\ 2 \end{bmatrix}$$

that is, all vectors that can be written as

$$a\,\underline{u_1} + b\,\underline{u_2} + c\,\underline{u_3}$$

So if $\underline{w}$ is in the column space of $\underline{\underline{A}}$, there are numbers a, b, c such that

$$\underline{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 3 \\ 5 \\ 5 \end{bmatrix} + c \begin{bmatrix} 3 \\ 9 \\ -2 \end{bmatrix}$$

In matrix form, this is

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & -9 \\ 2 & 5 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

So that $\left\{ \begin{bmatrix} 3/5 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1/5 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for the column space of $\underline{\underline{A}}$

Note: These are not the bases given in the back of the book, which is fine since a basis is not unique.