Quiz 2 Key

Math 320, Lecture 2 - Spring 2009

To get the most of this quiz, allow yourself no more than 20 minutes to completely answer it, and do not use any notes or outside help. I will grade it if you hand it in during discussion on February 16 (for Monday sections), February 18 (for Wednesday sections).

1. Problem 1 (10 points)
   An epidemic hits our favorite mammal to model: bunnies. As a result, the bunnies can no longer reproduce. The death rate is proportional to the population of the remaining bunnies.
   (a) Write the differential equation for the population P(t) of rabbits at time t.
   (b) Assuming that $P(0) = P_0$, solve the initial value problem.
   (c) Suppose that when the epidemic hits at t=0, there are 1500 rabbits. Six months after crisis strikes, there are only 500 rabbits remaining. How many rabbits will be alive in 12 months (t=12)?

   Solution:

   This problem was meant to use remind you of the logistic equation problems we did where $\frac{dP}{dt} = (\beta - \delta)P$, where β and δ are the rate of births and of deathes, respectively. Recall homework problem # 17 from Section 2.2.

   (a) The general equation is $\frac{dP}{dt} = (\beta - \delta)P$, where β and δ are the birth and death rates, respectively. In our problem, $\beta = 0$ and $\delta = kP$ for some positive constant k, because there are no births and the death rate is propornal to P. So the differential equation is $\frac{dP}{dt} = -kP^2$.

   (b) We can solve this by using the separation of variables technique.

$$\begin{aligned} \frac{dP}{dt} &= -kP^2 \\ \frac{1}{P^2} dP &= -k\ dt \\ \int \frac{1}{P^2} dP &= \int (-k)\ dt \\ -\frac{1}{P} &= -kt + C \\ \frac{1}{P} &= kt + C' \\ P &= \frac{1}{kt + C'} \end{aligned}$$

   At time $t = 0$, we have $P(0) = P_0$. We must use this information in order to solve for our constant C'.

$$\begin{aligned} P(t) &= \frac{1}{kt + C'} \\ P_0 &= \frac{1}{k(0) + C'} \\ P_0 &= \frac{1}{C'} \\ C' &= \frac{1}{P_0} \end{aligned}$$

Thus, we get

$$\begin{aligned}
P(t) &= \frac{1}{kt + \frac{1}{P_0}} \\
&= \frac{P_0}{P_0kt + 1} \\
&= \frac{P_0}{1 + P_0kt}.
\end{aligned}$$

(c) We see from the given information that $P_0 = 1500$. Also, $P(6) = 500$. Using this piece of information, we can solve for the constant k.

$$\begin{aligned}
P(t) &= \frac{1500}{1 + 1500kt} \\
500 &= \frac{1500}{1 + 1500k(6)} \\
1 + 9000k &= 3 \\
9000k &= 2 \\
k &= \frac{1}{4500}
\end{aligned}$$

Thus,

$$\begin{aligned}
P(t) &= \frac{1500}{1 + 1500kt} \\
&= \frac{1500}{1 + 1500\left(\frac{1}{4500}\right)t} \\
&= \frac{1500}{1 + \frac{1}{3}t} \\
&= \frac{4500}{3 + t}.
\end{aligned}$$

We can now easily see that after 12 months, there will be $P(12) = 300$ bunnies remaining.