1. (20 points) Find positive numbers x and y, whose sum is 75, such that xy^2 is maximized.

$f(x,y) = xy^2$ to maximize (+2)

constraint: $x+y=75 \rightsquigarrow x+y-75=0$

so $g(x,y) = x+y-75$ (+2)

$F(x,y,\lambda) = xy^2 - \lambda(x+y-75)$

$= xy^2 - \lambda x - \lambda y + 75\lambda$ (+2)

(+3)
$$\begin{cases} F_x = y^2 - \lambda \\ F_y = 2xy - \lambda \\ F_\lambda = -x-y+75 \end{cases} \rightsquigarrow \left.\begin{array}{ll} ① & y^2 - \lambda = 0 \\ ② & 2xy - \lambda = 0 \\ ③ & -x-y+75=0 \end{array}\right\}$$
(+1)

(+8)

$\left.\begin{array}{l} ①: \lambda = y^2 \\ ②: \lambda = 2xy \end{array}\right\}$ $y^2 = 2xy$ $y(y-2x)=0$

$y^2 - 2xy = 0$

So either $y=0$, but we want positive numbers so we discard this solution, or $y=2x$.

③: $-x-(2x)+75=0 \rightarrow 3x=75$

$x=25$

So $y = 2 \cdot 25 = 50$.

(+2) The two numbers are 25 and 50.

2. (20 points) Find the Taylor series of the function

$$f(x) = \ln(1 - 5x)$$

and find its radius of convergence.

This is $\ln(1 + (-5x))$ +2

so the taylor series is

$$(-5x) - \frac{(-5x)^2}{2} + \frac{(-5x)^3}{3} + \dots + \frac{(-1)^n(-5x)^{n+1}}{n+1} + \dots$$

$$= -5x - \frac{25x^2}{2} - \frac{5^3x^3}{3} + \dots + \frac{(-1)^n(-1)^{n+1}5^{n+1}x^{n+1}}{n+1} + \dots$$

$$= -5x - \frac{25x^2}{2} - \frac{5^3x^3}{3} + \dots + \frac{(-1)^{2n+1}5^{n+1}x^{n+1}}{n+1} + \dots$$

$$= -5x - \frac{25x^2}{2} - \frac{5^3x^3}{3} - \dots - \frac{5^{n+1}x^{n+1}}{n+1} - \dots$$

+12

The interval of convergence is

$$-1 < -5x \leq 1 \quad +3$$

$$\frac{1}{5} > x \geq -\frac{1}{5} \quad +3$$

there are 5 points for the Taylor series of $\ln(1+x)$
3 points for plugging in $(-5x)$
4 points for simplifying and getting the answer

3. (20 points) The rate of a continuous money flow starts at \$1000 and increases exponentially at 5% per year for 3 years. Find the accumulated amount of money flow if the interest earned is 11% compounded continuously.

+2

money flow: $f(x) = 1000e^{0.05x}$, $r = 0.11$ +2

$t = 3$ +2

formula: $e^{rt}\int_0^t f(x)e^{-rx}\,dx$ +2

+4 $= e^{0.11\cdot 3}\int_0^3 1000e^{0.05x}e^{-0.11x}\,dx$

+4 $= e^{0.33}\int_0^3 1000e^{-0.06x}\,dx$

+2 $= 1000e^{0.33}\left(\dfrac{e^{-0.06x}}{-0.06}\Big|_0^3\right.$

+2 $= \dfrac{1000e^{0.33}}{-0.06}\left(e^{-0.18} - 1\right)$

4. (20 points) Consider the differential equation

$$\frac{dy}{dx} - 2xy - 4x = 0; \qquad y(1) = 20.$$

$u = -x^2$
$du = -2x\,dx$ (+3)

a) Solve this differential equation.

$\frac{dy}{dx} - 2xy = 4x \rightarrow$ linear

(+3) $\int -2x\,dx = -x^2$
$I(x) = e^{-x^2}$

$e^{-x^2}\frac{dy}{dx} - 2xe^{-x^2}y = 4xe^{-x^2}$

(+3) $\frac{d}{dx}(e^{-x^2}y) = 4xe^{-x^2}$

$e^{-x^2}y = \int 4xe^{-x^2}dx$

$e^{-x^2}y = 2\int e^u(-du)$
$e^{-x^2}y = -2e^u + C$
$e^{-x^2}y = -2e^{-x^2} + C$
$y = -2 + Ce^{x^2}$ (+3)

$20 = -2 + Ce$
$C = \frac{22}{e}$ (+1)

so $\boxed{y = -2 + 22e^{x^2-1}}$ (+1)

b) Apply one step of Euler's method with step size 0.1 to approximate the value of y at $x = 1.1$.

$\frac{dy}{dx} = 2xy + 4x \qquad f(x,y) = 2xy + 4x$ (+2)

$y_1 = 20 + 0.1(2\cdot1\cdot20 + 4\cdot1)$
$= 20 + 0.1(44)$
$= 20 + 4.4$
$= 24.4$

(+2) formula
(+2) plugging in correctly

5. (20 points) Compute the following limit:

$$\lim_{x\to 0} \frac{5e^x - 5}{x^3 - 8x^2 + 7x}$$

$\boxed{\frac{0}{0}}$ +5

$$\overset{HR}{=} \underbrace{\lim_{x\to 0} \frac{5e^x}{3x^2 - 16x + 7}}_{+10} = \frac{5}{7}$$ +5