OSCULATION POINTS AT THE CUSPS OF DRINFELD MODULAR CURVES

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Abstract. The study of Weierstrass points on curves defined over fields of positive characteristic is fraught with difficulties, and as such little is known about them. In this paper we adapt an argument of Atkin's on classical modular curves to the Drinfeld setting. More precisely, we exhibit an infinite family of curves $X_0(\mathfrak{n})$, for $\mathfrak{n}$ an ideal in $\mathbb{F}_q[T]$, such that the cusp at infinity is an osculation point of the curve.

1. Introduction and Statement of Results

Given a smooth irreducible projective curve of genus $g \geq 2$ defined over an algebraically closed field of characteristic 0, we say that a point P on X is a *Weierstrass point* if there is a nonzero rational function F on X with a pole of order less than or equal to g at P and regular everywhere else. In this case, the set of such points is non-empty and finite.

Because of the geometric significance of such points, given a curve of arithmetic import it is natural to study its Weierstrass points. Such work was done for three families that are important to number theorists: the Fermat curves, and the modular curves $X(N)$ and $X_0(N)$. The interested reader should see Rohrlich's paper [13] for a concise account of some of the early results obtained in these cases, the most important of which are due to Atkin, Hasse, Lehner and Newman, Ogg, Petersson, and Schoeneberg. Then in 1985 Rohrlich [14] computed a modular form for $\mathrm{SL}_2(\mathbb{Z})$ whose divisor encodes information about the reduction modulo ℓ of the Weierstrass points of $X_0(\ell)$, for ℓ a prime. Building on these results, later work of Ahlgren and Ono [1] showed that not only were the elliptic curves underlying the Weierstrass points of $X_0(\ell)$ supersingular at ℓ, which was a result already obtained by Ogg [12], but furthermore that

$$\prod_{Q \in X_0(\ell)} (x - j(Q))^{\mathrm{wt}(Q)} \equiv \prod_{\substack{E/\overline{\mathbb{F}}_\ell \\ E \text{ supersingular}}} (x - j(E))^{g_\ell(g_\ell - 1)} \pmod{\ell},$$

where the quantity $\mathrm{wt}(Q)$ is a non-negative integer which is positive if and only if Q is a Weierstrass point, and g_ℓ is the genus of $X_0(\ell)$.

The situation where the curve is defined over a field of positive characteristic is more complicated: It can be the case that for each point P there exists a nonzero rational function with a pole of order less than or equal to the genus of the curve at P and regular elsewhere. Accordingly, to ensure that the set of Weierstrass points be finite, a modified definition of Weierstrass points must be used, which will be given below. Instead, in this setting we call *osculation points* the points P such that there exists a nonzero rational function with a pole of order less than or equal to the genus of the curve at P and regular elsewhere. Under certain circumstances – when the curve is said to have a *classical gap sequence* – the osculation points and the Weierstrass points of the curve coincide, as is the case in characteristic 0.

Recently, Baker [4] considerably generalized Ogg's result concerning the reduction modulo ℓ of Weierstrass points on $X_0(\ell)$ which we mentioned above, and showed that under certain hypotheses the Weierstrass points of a curve defined over a local field must correspond to singular points on the special fiber. His argument uses a Specialization Lemma which, to speak roughly, allows one to transport information from the dual graph associated to a regular semistable model of the curve to the curve itself. Notably, this Specialization Lemma applies to the characteristic 0 as well as the positive characteristic case. In addition, it allows one to show that a tropical curve of genus $g \geq 2$ has a Weierstrass point.

In this paper, we study conditions under which one can guarantee that the cusps 0 and ∞ on the Drinfeld modular curve $X_0(\mathfrak{n})$ are osculation points, following an argument presented by Atkin in the classical case [3]. This adds to the few results that are known about Weierstrass points and osculation points on Drinfeld modular curves: Following from the work of Baker we know that, as in the classical case, whenever $\mathfrak{n}$ is a prime ideal then the Drinfeld module underlying a Weierstrass point of $X_0(\mathfrak{n})$ is supersingular at $\mathfrak{n}$. A consequence of this fact is that if $\mathfrak{n}$ is prime, then the cusps are never Weierstrass points. (This result is also obtained by Armana [2], who shows that the cusps are not osculation points, from which it follows easily that they are not Weierstrass points.) In forthcoming work [15], the author has obtained, under certain somewhat severe hypotheses, a generalization of results obtained by Rohrlich in [14] on Weierstrass points for the curve $X_0(\mathfrak{n})$ for $\mathfrak{n}$ a prime ideal, which should prove to be an useful step towards studying how Weierstrass points are distributed in the fibers of the reduction map above supersingular points.

To state our result, we will need the following notation: Let $q \in \mathbb{Z}$ be a power of a prime, $\mathfrak{n}$ be an ideal of $\mathbb{F}_q[T]$, and write $\mathfrak{n} = \prod_{1 \leq i \leq s} \mathfrak{p}_i^{r_i}$ for the factorization of $\mathfrak{n}$ into prime ideals, with each prime ideal $\mathfrak{p}_i$ generated by a monic prime polynomial of degree d_i and each $r_i \geq 1$. Further, write $q_i = q^{d_i}$ for simplicity. Now let

$$\kappa(\mathfrak{n}) = \prod_{1 \leq i \leq s} \left(q_i^{\lfloor r_i/2 \rfloor} + q_i^{\lfloor (r_i-1)/2 \rfloor} \right),$$

and put $r(\mathfrak{n}) = 1$ if all of the r_i's are even and $r(\mathfrak{n}) = 0$ otherwise. Then we have:

Theorem 1. *Let $\mathfrak{p}$ be a prime ideal of $\mathbb{F}_q[T]$ and $\mathfrak{n}$ be an ideal of $\mathbb{F}_q[T]$. Then a sufficient condition for the cusps 0 and ∞ of $X_0(\mathfrak{p}^2\mathfrak{n})$ to be osculation points is:*

- $q \geq 3$ *and* $\kappa(\mathfrak{n}\mathfrak{p}) \geq 6(q-1)$ *when* $r(\mathfrak{p}\mathfrak{n}) = 0$ *and* $r(\mathfrak{p}^2\mathfrak{n}) = 1$*;*
- $\kappa(\mathfrak{n}\mathfrak{p}) \geq 6(q-1)$ *for all other values of* $r(\mathfrak{p}\mathfrak{n})$ *and* $r(\mathfrak{p}^2\mathfrak{n})$*.*

Remark. As with Atkin's result, this condition forces the product $\mathfrak{p}^2\mathfrak{n}$ to be "highly composite". To illustrate this, we give a few examples of $(\mathfrak{n}, \mathfrak{p})$ that satisfy the condition $\kappa(\mathfrak{n}\mathfrak{p}) \geq 6(q-1)$. Write $\mathfrak{p}^2\mathfrak{n} = \mathfrak{p}^m\mathfrak{m}$ for $(\mathfrak{p}, \mathfrak{m}) = 1$, and let p generate the ideal $\mathfrak{p}$. Then the condition on $\kappa(\mathfrak{p}\mathfrak{n})$ is satisfied in the following cases:

- With no restriction on $\mathfrak{m}$ if
  - $q \geq 5$ when p is of degree 2 or for any q if p if of degree ≥ 3 when $m = 3$, or
  - p is of degree ≥ 2 when $m \geq 4$, or
  - whenever $m \geq 5$;
- With no restriction on $\mathfrak{p}$ if
  - $\mathfrak{q}^4 \mid \mathfrak{m}$ for $\mathfrak{q}$ a prime ideal distinct from $\mathfrak{p}$ (similarly as above, a lower power is enough if $\mathfrak{q}$ is generated by a polynomial of high enough degree),

– $\mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3^2 \mid \mathfrak{m}$ for $\mathfrak{q}_1$, $\mathfrak{q}_2$, and $\mathfrak{q}_3$ distinct prime ideals,
– $\mathfrak{m}$ is divisible by s distinct primes with $2^s \geq 3(q-1)$.

As remarked above, if one could show that the curves $X_0(\mathfrak{n})$ have a classical gap sequence, this result would imply that the cusps 0 and ∞ are Weierstrass points of $X_0(\mathfrak{n})$ when $\mathfrak{n}$ is highly composite, which would present a nice counterpoint to the result that they are never Weierstrass points when $\mathfrak{n}$ is prime.

Acknowledgements

The author would like to thank Felipe Voloch who kindly pointed out an inaccuracy in the original version of this paper, as well as the referee of an earlier version of this paper for his helpful comments and suggestions.

2. Preliminaries

As before we fix q a power of a prime, and denote by $\mathbb{F}_q$ the finite field with q elements. We will denote by A the ring of polynomials in an indeterminate T, $A = \mathbb{F}_q[T]$, and by K the field $\mathbb{F}_q(T)$, the field of fractions of A. Then for $x \in K$ we may define the degree valuation $v_\infty(x) = \deg(x)$ associated to the infinite place of K. We will write $K_\infty = \mathbb{F}_q((1/T))$ for the completion of K at its infinite place, and

$$C = \hat{\bar{K}}_\infty$$

for the completed algebraic closure of K_∞. Finally we will also need the set $\Omega = \mathbb{P}^1(C) - \mathbb{P}^1(K_\infty) = C - K_\infty$. It is possible to endow this set with the structure of a rigid analytic space, see for example [9] for a concise reference, and we will call this space the Drinfeld upper half-plane. The matrix group $\mathrm{GL}_2(A)$ acts on Ω via $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) z = \frac{az+b}{cz+d}$.

We now briefly go over the definition of the curves $X_0(\mathfrak{n})$, and refer the reader to Gekeler's book [7] for details and proofs. For $\mathfrak{n}$ an ideal of A, define the congruence subgroup $\Gamma_0(\mathfrak{n})$ of $\mathrm{GL}_2(A)$ by

$$\Gamma_0(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(A) \mid c \equiv 0 \pmod{\mathfrak{n}} \right\}.$$

Then the set $\Gamma_0(\mathfrak{n})\backslash\Omega$ inherits a rigid analytic structure from Ω. Furthermore, Drinfeld shows in [5] that there exists a smooth irreducible affine curve defined over K, which we will denote by $Y_0(\mathfrak{n})$, such that the rigid analytic space associated to it (see [6] for a description of this space) is canonically isomorphic to $\Gamma_0(\mathfrak{n})\backslash\Omega$ as a rigid analytic space over C. The curve $Y_0(\mathfrak{n})$ has a unique smooth projective model which we denote by $X_0(\mathfrak{n})$. Throughout, we will think of $X_0(\mathfrak{n})$ as defined over C, and its points will be C-valued points.

As sets of C-valued points, $X_0(\mathfrak{n})$ is obtained from $Y_0(\mathfrak{n})$ by adding finitely many points called cusps, and these points are in one-to-one correspondence with the set $\Gamma_0(\mathfrak{n})\backslash\mathbb{P}^1(K)$. We can give an alternative description of this set: Write $\bar{\Gamma}_0(\mathfrak{n})$ for the image of $\Gamma_0(\mathfrak{n})$ in $\mathrm{GL}_2(A/\mathfrak{n})$, and $(A/\mathfrak{n})^2_{prim}$ for the set of vectors in $A/\mathfrak{n} \times A/\mathfrak{n}$ that span a nonzero direct summand of $A/\mathfrak{n} \times A/\mathfrak{n}$. Then the set of cusps of $X_0(\mathfrak{n})$ is in bijection with the set $\bar{\Gamma}_0(\mathfrak{n})\backslash(A/\mathfrak{n})^2_{prim}/\mathbb{F}_q^\times$, as shown in [8]. If $\mathfrak{n} \neq 1$, we will be particularly interested in the cusp given by the equivalence class of $(1,1) \in (A/\mathfrak{n})^2_{prim}$, which we will denote by 0, and the cusp given by the equivalence class of $(1,n) \in (A/\mathfrak{n})^2_{prim}$, where n is the monic generator of $\mathfrak{n}$, which we will denote by ∞.

In [10], Gekeler computes the genera of these curves: Recall our notation from the Introduction: we write $\mathfrak{n} = \prod_{1\le i\le s} \mathfrak{p}_i^{r_i}$ for the factorization of $\mathfrak{n}$ into prime ideals, with the generator of $\mathfrak{p}_i$ of degree d_i and each $r_i \geq 1$. Further, write $q_i = q^{d_i}$ for simplicity. We will need the quantities $\kappa(\mathfrak{n})$ and $r(\mathfrak{n})$ defined in the Introduction, as well as

$$\varepsilon(\mathfrak{n}) = \prod_{1\le i\le s} q_i^{r_i-1}(q_i+1).$$

Then we have

Proposition 2 (Gekeler [8]). *The genus of the curve $X_0(\mathfrak{n})$ is given by*

$$g(\mathfrak{n}) = 1 + \frac{\varepsilon(\mathfrak{n}) - (q+1)\kappa(\mathfrak{n}) - 2^{s-1}(r(\mathfrak{n})q(q-1) + (q+1)(q-2))}{q^2-1}. \tag{2.1}$$

The curve $X_0(\mathfrak{n})$ comes equipped with an involution denoted $W_\mathfrak{n}$, the Fricke involution. For $z \in \Omega$, $W_\mathfrak{n}$ send the equivalence class of z in $\Gamma_0(\mathfrak{n})\backslash\Omega$ to the equivalence class of $\frac{-1}{nz}$, where n is again a monic generator of the ideal $\mathfrak{n}$. For $(a,c) \in (A/\mathfrak{n})^2_{prim}$, $W_\mathfrak{n}$ send the equivalence class of (a,c) to the equivalence class of $(a, n/c)$. In particular, $W_\mathfrak{n}$ interchanges the two cusps 0 and ∞.

Finally, we review a few facts about the geometry of curves; proofs may be found in [11]. Let k be an algebraically closed field and X be a smooth irreducible projective curve over k of genus $g \geq 2$, and fix P a point of X. If there is a nonzero rational function F on X such that F has a pole of order exactly n at P and F is regular elsewhere, we say that n is a a *pole number* at P. Otherwise, if no such function exists, we say that n is a *gap* at P. There are exactly g gaps at P, and if $n_1(P), \ldots, n_g(P)$ are the gaps at P, indexed such that $n_i(P) < n_j(P)$ if $i < j$, we say that $(n_1(P), \ldots, n_g(P))$ is the gap sequence at P.

For a fixed curve X, it can be shown that there exists a sequence of positive integers $(n_1, \ldots, n_g)$ with $n_i < n_j$ if $i < j$ such that $(n_1, \ldots, n_g)$ is the gap sequence at P for all but finitely many points of X. We call this sequence the *canonical gap sequence* of X. The finitely many points that have a different gap sequence are called the *Weierstrass points* of X. Rephrasing the definition given in the Introduction, if a point P has gap sequence $(n_1(P), \ldots, n_g(P))$ and $n_g(P) > g$, then P is an osculation point of the curve.

We say that X has a classical gap sequence if its gap sequence is $(1, \ldots, g)$. This is always the case if k has characteristic 0. It is clear that if X has a classical gap sequence, then the osculation points are exactly the Weierstrass points; otherwise every point of X is an osculation point.

3. Proof of the Theorem

We start by remarking that if F is a nonzero rational function with a pole of order m at the cusp ∞ on $X_0(\mathfrak{n})$ and regular elsewhere, then the function $F \circ W_\mathfrak{n}$ is also a nonzero rational function, and it has a pole of order m at the cusp 0 and is regular elsewhere. The same assertion with 0 in place of ∞ and ∞ in place of 0 is also true. Therefore 0 and ∞ will simultaneously be or not be osculation points for $X_0(\mathfrak{n})$.

Throughout we will continue to write n for the unique monic polynomial generating the ideal $\mathfrak{n}$, and write $\mathfrak{p}$ for a prime ideal of A and p for its unique monic generator. We will also let $d = \deg p$.

Lemma 3. *Let $\mathfrak{p}$, $\mathfrak{n}$, p and n be as above. The index of $\Gamma_0(\mathfrak{p}^2\mathfrak{n})$ in $\Gamma_0(\mathfrak{p}\mathfrak{n})$ is q^d.*

Proof. This is easily shown by noticing that the set

$$\left\{ \begin{pmatrix} 1 & 0 \\ kpn & 1 \end{pmatrix} : k \in A, \deg k < \deg p \right\},$$

including $k = 0$, is a complete set of right coset representatives. The cardinality of this set is q^d. □

Each of these coset representatives fixes the cusp 0, and as a consequence it follows that the natural covering map $X_0(\mathfrak{p}^2\mathfrak{n}) \to X_0(\mathfrak{p}\mathfrak{n})$, which is of degree q^d, is fully ramified above the cusp 0 of $X_0(\mathfrak{p}\mathfrak{n})$. Thus we have

Lemma 4. *If F is a nonzero rational function on $X_0(\mathfrak{p}\mathfrak{n})$ with a pole of order m at the cusp 0, and regular elsewhere, then its pullback to $X_0(\mathfrak{p}^2\mathfrak{n})$ via the natural covering map has a pole of order $q^d m$ at 0 and is regular elsewhere on $X_0(\mathfrak{p}^2\mathfrak{n})$.*

Whether or not 0 is an osculation point for $X_0(\mathfrak{p}\mathfrak{n})$, there is certainly a nonzero rational function F on $X_0(\mathfrak{p}\mathfrak{n})$ with a pole of order less than or equal to $g(\mathfrak{p}\mathfrak{n}) + 1$ at 0 and regular elsewhere, since there are only $g(\mathfrak{p}\mathfrak{n})$ gaps at each point. Thus the pullback of F to $X_0(\mathfrak{p}^2\mathfrak{n})$ has a pole of order less than or equal to $q^d(g(\mathfrak{p}\mathfrak{n}) + 1)$ at 0 and is regular elsewhere. Therefore we have:

Lemma 5. *If $g(\mathfrak{p}^2\mathfrak{n}) \geq q^d(g(\mathfrak{p}\mathfrak{n})+1)$ then the cusps 0 and ∞ of $X_0(\mathfrak{p}^2\mathfrak{n})$ are osculation points.*

Using formula (2.1), a tedious but elementary computation determines that this happens in the cases listed in the statement of the theorem, and this completes the proof.

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