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DRINFELD MODULAR FORMS MODULO $\mathfrak{p}$

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Abstract. The classical theory of "modular forms modulo ℓ" was developed by Serre and Swinnerton-Dyer in the early 1970's. Their results revealed the important role that the quasi-modular form E_2, Ramanujan's Θ-operator, and the filtration of a modular form would subsequently play in applications of their theory. Here we obtain the analog of their results in the Drinfeld modular form setting.

1. Introduction and statement of results

In this paper we study the arithmetic properties of the coefficients of Drinfeld modular forms, using as our tools a derivation on the algebra of Drinfeld modular forms and knowledge about the algebra obtained from the reduction of Drinfeld modular forms of all weights and types modulo a prime ideal. Our study is motivated by analogous theorems by Serre [12] and Swinnerton-Dyer [13] (see also Chapter 2 of [11]) which give information about the interplay between Ramanujan's Θ-operator on q-series and the reduction of classical modular forms modulo a prime ℓ. The classical theory has numerous applications. Among them is the proof of Ramanujan's claim that the only so-called "Ramanujan-type" congruences for the partition function $p(n)$ are his famous congruences with modulus 5, 7, and 11, a proof given by Ahlgren and Boylan in [1]. The theory developed by Serre and Swinnerton-Dyer is also used by Elkies, Ono and Yang in [6] to determine a simple condition under which the meromorphic modular function $F(j(z))$, for $F(x) \in \mathbb{Z}[x]$ and $j(z)$ the classical j-function for $\mathrm{SL}_2(\mathbb{Z})$, satisfies $U(p)$ congruences modulo p, for p prime. These congruences generalize a classical result due to Lehner [10] that states that

$$j(z) \,|\, U(p) \equiv 744 \pmod{p}$$

for every prime $p \leq 11$.

Central to the classical theory of modular forms are the Eisenstein series: For $k \geq 2$ an even integer and z in the complex upper-half plane, define the Eisenstein series of weight k by

$$E_k(z) \stackrel{\text{def}}{=} \frac{1}{2\zeta(k)} \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{1}{(mz+n)^k}, \tag{1.1}$$

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where $\zeta(s)$ is the Riemann-zeta function. It is a fundamental fact that E_k is a modular form for $\mathrm{SL}_2(\mathbb{Z})$ of weight k if $k \geq 4$. For $k = 2$, the convergence of the series is not absolute so that the order of summation must be specified: E_2 is defined to be the double sum above summing first over n for fixed m and then summing over m. Even with this modification E_2 is not modular; instead it satisfies a slightly more complicated transformation rule. Despite this E_2 still plays an important role, as we will see later. For each $k \geq 2$ even, we have $E_k(z+1) = E_k(z)$, and so E_k has a Fourier expansion, and setting $q = e^{2\pi i z}$, we will call this series its q-expansion.

Ramanujan defined the Θ-operator on modular forms by

$$\Theta \stackrel{\text{def}}{=} \frac{1}{2\pi i}\frac{d}{dz} = q\frac{d}{dq},$$

and he showed the following identities:

$$\Theta(E_2) = \frac{E_2^2 - E_4}{12}, \quad \Theta(E_4) = \frac{E_2E_4 - E_6}{3}, \quad \Theta(E_6) = \frac{E_2E_6 - E_4^2}{2}.$$

Thus the algebra generated by E_2, E_4 and E_6, which contains all of the modular forms for $\mathrm{SL}_2(\mathbb{Z})$, is stable under the derivation Θ.

Let us now consider the function field $K = \mathbb{F}_q(T)$ (for q a power of a prime p) in the place of the rational numbers. Notice that here q is used with a different meaning than above, but no confusion should arise from this. We consider the completion $K_\infty = \mathbb{F}_q((1/T))$ of K at its infinite place and the completed algebraic closure of K_∞, which we will denote by C to emphasize the analogy with the complex numbers. Then we may define the "Drinfeld upper-half plane" by $\Omega = C - K_\infty$.

For k a positive integer and $z \in \Omega$, Goss defines in [9] an Eisenstein series of weight $q^k - 1$ for this function field as

$$g_k \stackrel{\text{def}}{=} (-1)^{k+1}\tilde{\pi}^{1-q^k} L_k \sum_{\substack{a,b\in\mathbb{F}_q[T] \\ (a,b)\neq(0,0)}} \frac{1}{(az+b)^{q^k-1}}, \tag{1.2}$$

where L_k is the least common multiple of all monics of degree k, so that

$$L_k = (T^q - T)\ldots(T^{q^k} - T),$$

and $\tilde{\pi}$ is the Carlitz period, which roughly plays the role of the constant $2\pi i$ and will be defined in the second section of this paper. These series converge and thus define rigid analytic functions on Ω. They should be considered the analogs of the modular Eisenstein series given in (1.1), and they can be shown to be modular (the definition of a Drinfeld modular form will be given in Section 2.1). As an analog of E_2 defined above, Gekeler in [7] introduces the following object:

$$E \stackrel{\text{def}}{=} \frac{1}{\tilde{\pi}} \sum_{\substack{a\in\mathbb{F}_q[T] \\ a \text{ monic}}} \left(\sum_{b\in\mathbb{F}_q[T]} \frac{a}{az+b} \right).$$

Like E_2, E is not modular but satisfies a similar transformation rule under the action of $GL_2(\mathbb{F}_q[T])$. Finally, just as in the classical setting, we may define a uniformizing parameter at infinity which will be denoted in this paper by u. Its definition involves the Carlitz period mentioned above, as well as a certain lattice function which should be considered the analog of the exponential function. These objects are introduced in the second section of this paper and the precise definition of u follows.

The last series which is important in this paper is the Poincaré series of weight $q+1$ and type 1, first defined by Gerritzen and van der Put in [8, page 304]. Let H be the subgroup

$$\left\{\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}\right\} \subset \mathrm{GL}_2(\mathbb{F}_q[T])$$

and

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_q[T]).$$

Then we may define a series

$$h \stackrel{\text{def}}{=} \sum_{\gamma \in H \backslash \mathrm{GL}_2(\mathbb{F}_q[T])} \frac{\det \gamma \cdot u(\gamma z)}{(cz+d)^{q+1}}. \tag{1.3}$$

This is holomorphic provided that the sum converges and behaves well "at infinity", which can be shown using the properties of the function u. Furthermore, this series in fact defines a Drinfeld modular form.

The series g_1 and h generate the C-algebra of Drinfeld modular forms, just as E_4 and E_6 generate the $\mathbb{C}$-algebra of modular forms. We may again define the Θ-operator, this time by

$$\Theta \stackrel{\text{def}}{=} \frac{1}{\tilde{\pi}} \frac{d}{dz} = -u^2 \frac{d}{du}.$$

Now in analogy with the classical case, Gekeler [7] showed that

$$\Theta(E) = -E^2, \quad \Theta(g_1) = Eg_1 + h, \quad \Theta(h) = -Eh.$$

Hence the algebra generated by E, g_1 and h is stable under the derivation Θ.

It is well-known that the q-series expansions of the classical Eisenstein series for $k \geq 4$ positive and even can be given simply by

$$E_k = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where B_k is the kth Bernoulli number and $\sigma_{k-1}(n)$ is the sum of the $(k-1)$th powers of the divisors of n. Thus using the von Staudt-Clausen Theorem one can show that for every prime $\ell \geq 5$,

$$E_{\ell-1} \equiv 1 \pmod{\ell}$$

and

$$E_2 \equiv E_{\ell+1} \pmod{\ell},$$

where the congruence here is taken to mean the congruence modulo ℓ of corresponding coefficients in the q-series expansion. Serre and Swinnerton-Dyer showed that in fact, the relation $E_{\ell-1} \equiv 1 \pmod{\ell}$ is the only non-trivial relation modulo ℓ. In [7, Corollary 6.12], Gekeler proved that for $\mathfrak{p}$, the ideal generated by a monic prime polynomial of degree d, we have

$$g_d \equiv 1 \pmod{\mathfrak{p}},$$

where g_d is as in (1.2). Moreover, just as in the classical case, this is the only relationship modulo $\mathfrak{p}$. Thus g_d plays the role of the form $E_{\ell-1}$ in this setting. However, until now there had not been a clear analog to the second congruence written above. In light of the theorems proven in this paper, the analogous statement is

Theorem 1.1. *Let $\partial(g_d) = \Theta(g_d) - Eg_d$ (note that this can be shown to be modular). If $\mathfrak{p}$ is the ideal generated by a monic prime polynomial of degree d, then*

$$E \equiv -\partial(g_d) \pmod{\mathfrak{p}}.$$

Remark. Notice that in the classical case we have that

$$12\left(\Theta(E_{\ell-1}) - \frac{(\ell-1)}{12}E_{\ell-1}E_2\right)$$

is a weight $\ell+1$ modular form and

$$12\left(\Theta(E_{\ell-1}) - \frac{(\ell-1)}{12}E_{\ell-1}E_2\right) \equiv E_2 \pmod{\ell}.$$

In the classical case, Swinnerton-Dyer [13] showed that while the formal q-series $\Theta(f)$ for f a modular form is not a modular form itself, it is congruent to a modular form modulo ℓ for every prime ℓ. He also described how the application of the Θ-operator affects the *filtration* of a modular form, where the filtration of a q-series is defined to be the smallest integer k for which there is a modular form of weight k that is congruent to the series modulo ℓ. Of course, we may define the filtration of a Drinfeld modular form f (or more generally a u-series) for a prime ideal $\mathfrak{p}$ in the same manner, and this number will be denoted by $w_\mathfrak{p}(f)$. As in the classical case, we will say that $w_\mathfrak{p}(f) = -\infty$ if $f \equiv 0 \pmod{\mathfrak{p}}$. The first goal of this paper is to derive properties of filtrations in the Drinfeld setting. We prove the following:

Theorem 1.2. *Let f be a Drinfeld modular form of weight k and type l, and let $\mathfrak{p}$ be an ideal generated by a monic prime polynomial of degree d. If f has $\mathfrak{p}$-integral u-coefficients and is not identically zero modulo $\mathfrak{p}$, then the following are true:*

1. *$\Theta(f)$ is the reduction of a modular form modulo $\mathfrak{p}$.*
2. *We have $w_\mathfrak{p}(\Theta(f)) \equiv w_\mathfrak{p}(f) + 2 \pmod{q^d - 1}$ (where we take this to be vacuously true if $w_\mathfrak{p}(\Theta(f)) = -\infty$). Furthermore $w_\mathfrak{p}(\Theta(f)) \leq w_\mathfrak{p}(f) + q^d + 1$ with equality if and only if $w_\mathfrak{p}(f) \not\equiv 0 \pmod{p}$.*

Remark. The reader familiar with the theory in characteristic zero will remember that applying the Θ-operator usually increases the filtration of a modular form by $\ell + 1$. The only exception is when the filtration of the form is congruent to zero modulo ℓ, in which case the filtration decreases. In the Drinfeld modular setting the filtration must be congruent to zero modulo p (where we recall that p is the characteristic of $\mathbb{F}_q[T]$). Although something like this was to be expected, since the filtration is an integer and thus its reduction modulo an ideal of $\mathbb{F}_q[T]$ does not make sense, it does completely change the flavor of the theory.

Because Θ acts as $-u^2\frac{d}{du}$ on formal u-series, Θ necessarily acts nilpotently (Θ^p is identically zero on all forms), which is not the case in characteristic zero. Some knowledge of the exact "degree of nilpotency" of Θ on a particular form f may be obtained from the following theorem:

Theorem 1.3. *Define for every positive integer k and for p the characteristic of $\mathbb{F}_q[T]$ the integer $n(k,p)$ as the unique integer $0 \leq n(k,p) < p$ such that $k + n(k,p) \equiv 0 \pmod{p}$. Let f be a Drinfeld modular form of weight k and type l, and let $\mathfrak{p}$ be any ideal generated by a monic prime polynomial of degree d. If f has $\mathfrak{p}$-integral u-coefficients and is not identically zero modulo $\mathfrak{p}$, then*

$$w_\mathfrak{p}(\Theta^i(f)) = w_\mathfrak{p}(f) + i(q^d + 1) \qquad \textit{for } 0 \leq i \leq n(w_\mathfrak{p}(f), p).$$

Upon another iteration of the Θ-operator, we show that the filtration decreases, and a more precise statement of this theorem given in Section 3.2 gives a modular form to which $\Theta^{n(w_{\mathfrak{p}}(f),p)+1}(f)$ is congruent modulo $\mathfrak{p}$.

The last section of this paper presents applications of this theorem. Two of these applications make use of the fact that if we write a Drinfeld modular form f as

$$f = \sum_{i=0}^{\infty} a_i u^i,$$

applying Θ^j annihilates all a_i's such that $i \equiv 0, -1, \ldots, -j+1 \pmod{p}$. Thus by studying the application of iterations of Θ on a Drinfeld modular form we can under certain circumstances determine which classes modulo p contain coefficients that are non-zero, or zero, depending on the application, modulo $\mathfrak{p}$.

Remark. In a paper defining a Cohen bracket for Drinfeld modular forms, Uchino and Satoh [14] define a continuous higher derivation $\{D_n\}_{n=0}^{\infty}$ such that $D_1 = \Theta$ and D_p is not identically zero. For this application, as well as for others such as Bosser and Pellarin's study of Drinfeld quasi-modular forms in [2] and [3], this system of divided derivatives is the correct analog in characteristic p to Ramanujan's Θ-operator. However, their operators are not suitable for the study of modular forms modulo a prime ideal since they do not preserve $\mathfrak{p}$-integrality. It is possible that this is yet another instance in which an object that plays multiple roles in characteristic 0 has to be replaced by several different objects in characteristic p.

2. Preliminaries

As in the introduction, we will fix q a power of a prime p and denote by $\mathbb{F}_q$ the finite field with q elements. We will denote by A the ring of polynomials in an indeterminate T, $A = \mathbb{F}_q[T]$. From the introduction, we recall that $K = \mathbb{F}_q(T)$ is the field of fractions of A, $K_\infty = \mathbb{F}_q((1/T))$ is the completion of K at its infinite place,

$$C = \hat{\bar{K}}_\infty$$

is the completed algebraic closure of K_∞, and $\Omega = C - K_\infty$ is the Drinfeld upper-half plane.

Let Λ be an A-lattice of C, by which we mean a finitely-generated A-submodule having finite intersection with each ball of finite radius contained in C. We will need in this paper the following lattice function:

$$e_\Lambda(z) \stackrel{\text{def}}{=} z \prod_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} \left(1 - \frac{z}{\lambda}\right).$$

It can be shown that the product converges uniformly on bounded sets in C, thus defining an entire (in the rigid analytic sense) surjective function on C.

If we fix an A-lattice Λ of rank r in C, then for every $a \in A$ there is a unique map ϕ_a^Λ such that for all $z \in C$,

$$\phi_a^\Lambda(e_\Lambda(z)) = e_\Lambda(az).$$

The map

$$\phi^\Lambda : a \mapsto \phi_a^\Lambda$$

defines a ring homomorphism of A into the ring $\mathrm{End}_C(\mathbb{G}_a)$ of additive polynomials over C. $\mathrm{End}_C(\mathbb{G}_a)$ is the non-commutative ring of polynomials of the form

$$\sum a_i X^{p^i},$$

where multiplication is defined by composition. If we write $\tau = X^q$ and let $C\{\tau\} \subset \mathrm{End}_C(\mathbb{G}_a)$ be the subalgebra of $\mathrm{End}_C(\mathbb{G}_a)$ generated by τ, then in fact it can be shown that ϕ^Λ takes values in $C\{\tau\}$ and for $a \in A$ of degree d we have

$$\phi_a^\Lambda = \sum_{0 \le i \le rd} l_i \tau^i \tag{2.1}$$

with $l_0 = a$ and $l_{rd} \neq 0$.

A ring homomorphism $\phi : A \to C\{\tau\}$ that is given by (2.1) is called a *Drinfeld module of rank r* over C. The association $\Lambda \mapsto \phi^\Lambda$ is a bijection of the set of A-lattices of rank r in C with the set of Drinfeld modules of rank r over C.

An important Drinfeld module is Carlitz's module ρ of rank 1, first studied by Carlitz in [4] and [5], and defined by:

$$\rho_T = TX + X^q.$$

This Drinfeld module corresponds to a certain rank-1 A-lattice $L = \tilde{\pi}A$, where the "Carlitz period" $\tilde{\pi} \in K_\infty(\sqrt[q-1]{-T})$ is defined up to a $(q-1)$th root of unity. We choose one such $\tilde{\pi}$ and fix it for the remainder of this paper.

Now consider the function

$$u(z) \stackrel{\text{def}}{=} \frac{1}{e_L(\tilde{\pi}z)}$$

for $L = \tilde{\pi}A$ and $\tilde{\pi}$ the Carlitz period. Then we have that for any $c > 1$, u induces an isomorphism of the set

$$A \backslash \{z \in \Omega \mid \inf_{x \in K_\infty} |z - x| \geq c\}$$

with a pointed ball $B_r \backslash \{0\}$. Thus $u(z)$ can be used as a "parameter at infinity", analogously to $q = e^{2\pi i z}$ in the classical case.

2.1. **Drinfeld modular forms.** A function $f : \Omega \to C$ is called a *Drinfeld modular form of weight k and type l*, where $k \geq 0$ is an integer and l is a class in $\mathbb{Z}/(q-1)\mathbb{Z}$ if

(1) for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(A)$, $f(\gamma z) = (\det \gamma)^{-l}(cz+d)^k f(z)$;
(2) f is rigid analytic;
(3) f has an expansion $f(z) = F(u(z))$ where F is a power series with a positive radius of convergence.

From now on we will denote g_1, the Eisenstein series of weight $q-1$, simply by g. It is a Drinfeld modular form of weight $q-1$ and type 0. The Poincaré series h defined in (1.3) is a Drinfeld modular form of weight $q+1$ and type 1. It is a well-known fact (see for example [7]) that the graded C-algebra of Drinfeld modular forms of all weights and all types, denoted here by M, is the polynomial ring $C[g, h]$ (where each Drinfeld modular form corresponds to a unique isobaric polynomial).

We recall from the introduction the operator $\Theta = \tilde{\pi}^{-1}\frac{d}{dz} = -u^2\frac{d}{du}$ and further define $\partial_k = \Theta + kE$ on the C-vector space of forms of weight k and type l; we will simply write ∂ when the weight is implicit or when we wish to consider ∂ as a differential operator of weight 2 on the graded algebra of Drinfeld modular forms M. We collect here a series of well-known results about these operators:

Lemma 2.1 (see for example [7]). *Let f_i be a Drinfeld modular form of weight k_i and type l_i, for $i = 1, 2$.*

1. *$\partial_{k_i}(f_i)$ is again a Drinfeld modular form, of weight $k_i + 2$ and type $l_i + 1$.*
2. *If $k = k_1 + k_2$, then $\partial_k(f_1 f_2) = \partial_{k_1}(f_1) f_2 + f_1 \partial_{k_2}(f_2)$.*
3. *$\partial(g) = h$ and $\partial(h) = 0$.*
4. *$\partial^2(g_d) = 0$ for each d.*

2.2. **Modular forms modulo $\mathfrak{p}$.** From now on we will fix a monic prime polynomial in A of degree d and denote by $\mathfrak{p}$ the principal ideal that it generates. The reduction homomorphism $A \to \mathbb{F}_{\mathfrak{p}} \overset{\text{def}}{=} A/\mathfrak{p}$ and everything derived from it will be denoted by a tilde $a \mapsto \tilde{a}$. Let $A_{\mathfrak{p}}$ be the localization of A at $\mathfrak{p}$, let $M_{\mathfrak{p}}$ denote the ring of modular forms having coefficients in K with denominators prime to $\mathfrak{p}$, and write

$$\tilde{M} \overset{\text{def}}{=} \{\tilde{f} \in \mathbb{F}_{\mathfrak{p}}[[u]] \mid \exists f \in M_{\mathfrak{p}} \text{ such that } f \equiv \tilde{f} \pmod{\mathfrak{p}}\}$$

(where as before $f_1 \equiv f_2 \pmod{\mathfrak{p}}$ means the congruence modulo $\mathfrak{p}$ of corresponding coefficients in the u-series expansion) for the $\mathbb{F}_{\mathfrak{p}}$-algebra of Drinfeld modular forms modulo $\mathfrak{p}$.

Following [13], we find it convenient to adopt the following notation: If f is a function which has a u-series expansion $\sum_{i=0}^{\infty} a_i u^i$ such that every a_i is in $A_{\mathfrak{p}}$, then $\tilde{f}$ will denote the formal power series $\sum_{i=0}^{\infty} \tilde{a}_i u^i$. Similarly, if $\phi(X, Y)$ is a polynomial in $A_{\mathfrak{p}}[X, Y]$, then $\tilde{\phi}(X, Y)$ will denote the polynomial in $\mathbb{F}_{\mathfrak{p}}[X, Y]$ obtained from ϕ by reducing its coefficients modulo $\mathfrak{p}$. Naturally we will wish to evaluate these polynomials at the formal power series in u corresponding to $\tilde{g}$ and $\tilde{h}$ and denote by $\tilde{\phi}(\tilde{g}, \tilde{h})$ the element of $\mathbb{F}_{\mathfrak{p}}[[u]]$ obtained from this polynomial by substitution. As a consequence of this notation, if f is a Drinfeld modular form in $M_{\mathfrak{p}}$, there is a unique polynomial ϕ such that $f = \phi(g, h)$, and $\tilde{f} = \tilde{\phi}(\tilde{g}, \tilde{h})$. Finally, motivated by the derivation ∂ described above, we define a derivation, also denoted ∂, on $A_{\mathfrak{p}}[X, Y]$ and $\mathbb{F}_{\mathfrak{p}}[X, Y]$ by setting $\partial(X) = Y$ and $\partial(Y) = 0$ in both cases. The operator Θ described earlier analogously extends from $A_{\mathfrak{p}}[[u]]$ to $\mathbb{F}_{\mathfrak{p}}[[u]]$.

Since $M_{\mathfrak{p}}$ contains the elements g and h, we have the following composition of homomorphisms:

$$\begin{array}{rcl} A_{\mathfrak{p}}[X, Y] \xrightarrow{\sim} \mathbb{F}_{\mathfrak{p}}[X, Y] & \xrightarrow{\epsilon} & \mathbb{F}_{\mathfrak{p}}[[u]] \\ (X, Y) & \mapsto & (\tilde{g}, \tilde{h}) \end{array} \tag{2.2}$$

(where we recall that the tilde denotes the "reduction modulo $\mathfrak{p}$" homomorphism). Consequently we will assign weight $q-1$ to X and weight $q+1$ to Y. Here we quote a theorem from [7, Corollary 6.12] in order to make more precise a result about the reduction of g_d modulo $\mathfrak{p}$ stated in the introduction:

Theorem 2.2. *Let $A_d \in A[X, Y]$ be the polynomial defined by $A_d(g, h) = g_d$. Assuming the notation and hypotheses above, the following are true:*

1. *$\tilde{A}_d(X, Y)$ is square-free.*
2. *$\tilde{M} \cong \mathbb{F}_{\mathfrak{p}}[X, Y]/(\tilde{A}_d(X, Y) - 1)$.*

3. New results

As a consequence of Theorem 2.2, if $f_i \in M_{\mathfrak{p}}$ is of weight k_i for $i = 1$, 2 and $f_1 \equiv f_2 \pmod{\mathfrak{p}}$, then $k_1 \equiv k_2 \pmod{q^d - 1}$. Thus $\tilde{M}$ has a natural grading by $\mathbb{Z}/(q^d - 1)\mathbb{Z}$. As in the introduction we will denote by $w_{\mathfrak{p}}(f)$ the filtration of

f, which is defined to be the smallest integer k such that there exists a Drinfeld modular form of weight k congruent to f modulo $\mathfrak{p}$, with the convention that the form 0 has weight $-\infty$ as before.

3.1. **Flushing out the analogy.** To continue the analogy with the classical case, Theorem 1.1 indicates that the analog of the modular form $E_{\ell+1}$ is the Drinfeld modular form $\partial(g_d)$. The theorem below, which gives Theorem 1.1, shows that indeed $\partial(g_d)$ shares the important properties that $E_{\ell+1}$ enjoys.

Theorem 3.1. *Let $B_d \in A[X,Y]$ be the polynomial defined by $B_d(g,h) = \partial(g_d)$. Assuming the notation and hypotheses above, the following are true:*

1. *$\tilde{B}_d(X,Y)$ shares no common factor with $\tilde{A}_d(X,Y)$.*
2. *We have $E \equiv -\partial(g_d) \pmod{\mathfrak{p}}$.*

Proof. For the proof of the first fact, let a be an irreducible factor of $\tilde{A}_d$ over $\overline{\mathbb{F}}_{\mathfrak{p}}[X,Y]$ and write $\tilde{A}_d = a \cdot b$. Since $\tilde{A}_d$ is square-free, a does not divide b. We have

$$\tilde{B}_d = \partial(\tilde{A}_d) = \partial(a)b + a\partial(b)$$

and a divides $\tilde{B}_d$ if and only if a divides $\partial(a)$. Since a must be isobaric, we can have either $a = X$, $a = X^{q+1} + cY^{q-1}$ for some nonzero c in the algebraic closure of $\mathbb{F}_{\mathfrak{p}}$ or $a = Y$. In the first two cases, we have respectively that $\partial(a) = Y$ and $\partial(a) = X^q Y$, so a does not divide $\partial(a)$.

The third possibility (in which case a divides $\partial(a)$) does not happen. In other words, Y does not divide $\tilde{A}_d$ for any d. This can be shown using induction on d and the recursive formula, proven in [7, Proposition 6.9],

$$\tilde{A}_d = \tilde{A}_{d-1} X^{q^{d-1}} + (T^{q^{d-1}} - T)\tilde{A}_{d-2} Y^{q^{d-2}(q-1)} \tag{3.1}$$

with $\tilde{A}_0 = 1$ and $\tilde{A}_1 = X$. By (3.1), if Y does not divide $\tilde{A}_{d-1}$, then Y does not divide $\tilde{A}_d$. Obviously Y does not divide $\tilde{A}_1$, so Y does not divide $\tilde{A}_d$ for any d.

For the proof of the second fact, it suffices to note that since $g_d \equiv 1 \pmod{\mathfrak{p}}$, $\Theta(g_d) \equiv 0 \pmod{\mathfrak{p}}$ and $\partial(g_d) = \Theta(g_d) + (q^d - 1)Eg_d \equiv -E \pmod{\mathfrak{p}}$. □

We will also need a result on modular forms that have lower filtration than weight:

Proposition 3.2. *Let f be a Drinfeld modular form in $M_{\mathfrak{p}}$ of weight k and type l with $\tilde{f} \neq 0$, and write $f = \phi(g,h)$. Then $w_{\mathfrak{p}}(f) < k$ if and only if $\tilde{A}_d | \tilde{\phi}$.*

Proof. Suppose that f' is of weight strictly less than f and $f \equiv f' \pmod{\mathfrak{p}}$. Write $f' = \psi(g,h)$ with $\psi \in A_{\mathfrak{p}}[X,Y]$. Then

$$\tilde{\phi} = c(\tilde{A}_d - 1) + \tilde{\psi}$$

for some polynomial $c \in \mathbb{F}_{\mathfrak{p}}[X,Y]$. Writing $c = \sum_{i=0}^n c_i$ as a sum of its isobaric components with c_i of weight strictly less than c_{i+1}, we have that

$$\tilde{\phi} = c_n \tilde{A}_d, \quad c_0 = \tilde{\psi} \quad \text{and} \quad c_i = c_{i-1}\tilde{A}_d \quad \text{for} \quad i = 1, \ldots n,$$

and $\tilde{A}_d$ divides $\tilde{\phi}$.

Suppose now that $\tilde{\phi} = \tilde{A}_d \tilde{\psi}$ for some polynomial $\tilde{\psi} \in \mathbb{F}_{\mathfrak{p}}[X,Y]$ which must be isobaric of weight $k - q^d + 1$. Lifting $\tilde{\psi}$ to $\psi \in A[X,Y]$, we have that $f' = \psi(g,h)$ is of weight strictly less than k and $f \equiv f' \pmod{\mathfrak{p}}$. □

3.2. **Proofs of theorems.** We are now ready to prove the theorems stated in the introduction.

Proof of Theorem 1.2. By Lemma 3.1, $\Theta(f) \equiv \partial(f)g_d + k\partial(g_d)f \pmod{\mathfrak{p}}$, which is a form of weight $k + q^d + 1$ and type $l + 1$. Now without loss of generality assume that f is of weight $w_{\mathfrak{p}}(f)$. Since $\Theta(f)$ is congruent to a form of weight $w_{\mathfrak{p}}(f) + q^d + 1$ it follows that

$$w_{\mathfrak{p}}(\Theta(f)) \equiv w_{\mathfrak{p}}(f) + 2 \pmod{q^d - 1}.$$

Furthermore, since f is of weight $w_{\mathfrak{p}}(f)$, then $\tilde{A}_d$ does not divide $\tilde{\phi}$. We have that

$$\Theta(\tilde{f}) = \partial(\tilde{\phi}(\tilde{g}, \tilde{h}))\tilde{A}_d(\tilde{g}, \tilde{h}) + w_{\mathfrak{p}}(f)\tilde{B}_d(\tilde{g}, \tilde{h})\tilde{\phi}(\tilde{g}, \tilde{h})$$

so that $\Theta(\tilde{f})$ is the image in $\mathbb{F}_{\mathfrak{p}}[[u]]$ of the polynomial

$$\partial(\tilde{\phi})(X, Y)\tilde{A}_d(X, Y) + w_{\mathfrak{p}}(f)\tilde{B}_d(X, Y)\tilde{\phi}(X, Y)$$

under the map ϵ given in (2.2). Since $\tilde{A}_d$ and $\tilde{B}_d$ have no common factors, $\tilde{A}_d$ divides $\partial(\tilde{\phi})\tilde{A}_d + w_{\mathfrak{p}}(f)\tilde{B}_d\tilde{\phi}$ if and only if $w_{\mathfrak{p}}(f) \equiv 0 \pmod{p}$. □

We now characterize the action of iterations of the operator Θ on the filtration of modular forms:

Proof of Theorem 1.3. Let f' be of weight $w_{\mathfrak{p}}(f)$ such that $f \equiv f' \pmod{\mathfrak{p}}$. As promised in the introduction, in addition to Theorem 1.3, we will show that

$$\Theta^{n(w_{\mathfrak{p}}(f),p)+1}(f) \equiv \partial^{n(w_{\mathfrak{p}}(f),p)+1}(f') \pmod{\mathfrak{p}}.$$

If $n(w_{\mathfrak{p}}(f), p) = 0$, the theorem is trivial and we have

$$\Theta(f) \equiv \Theta(f') \equiv \partial(f')g_d + w_{\mathfrak{p}}(f)\partial(g_d)f' \equiv \partial(f') \pmod{\mathfrak{p}},$$

thus proving the additional assertion.

Suppose now that $0 < n(w_{\mathfrak{p}}(f), p) < p$. We define a sequence of modular forms in the following manner, for $0 \leq i \leq n(w_{\mathfrak{p}}(f), p) + 1$:

$$\begin{aligned} f_0 &= f' \\ f_1 &= \partial(f')g_d + w_{\mathfrak{p}}(f)\partial(g_d)f' \\ f_2 &= \partial(f_1)g_d + (w_{\mathfrak{p}}(f) + 1)\partial(g_d)f_1 \\ &\ \vdots \\ f_i &= \partial(f_{i-1})g_d + (w_{\mathfrak{p}}(f) + i - 1)\partial(g_d)f_{i-1} \\ &\ \vdots\,. \end{aligned}$$

We first claim that

$$f_i \equiv \Theta^i(f) \pmod{\mathfrak{p}} \quad \text{for all } 0 \leq i \leq n(w_{\mathfrak{p}}(f), p) + 1.$$

This follows easily since for any Drinfeld modular form of weight k,

$$\Theta(f) \equiv \partial(f)g_d + k\partial(g_d)f \pmod{\mathfrak{p}}.$$

From this fact, since the weight of each f_i is

$$w_{\mathfrak{p}}(f) + i(q^d + 1) \equiv w_{\mathfrak{p}}(f) + i \pmod{p},$$

it follows that $\Theta(f_i) \equiv f_{i+1} \pmod{\mathfrak{p}}$. It suffices now to note that $f_1 \equiv f_2 \pmod{\mathfrak{p}}$ implies $\Theta(f_1) \equiv \Theta(f_2) \pmod{\mathfrak{p}}$.

Now since $f_i \equiv \Theta^i(f)$, of course $w_\mathfrak{p}(\Theta^i(f)) = w_\mathfrak{p}(f_i)$. For $1 \leq i \leq n(w_\mathfrak{p}(f), p)$, a simple induction shows that

$$w_\mathfrak{p}(f_{i-1}) = w_\mathfrak{p}(f) + (i-1)(q^d+1)$$

is not zero modulo p so that $w_\mathfrak{p}(f_i) = w_\mathfrak{p}(f) + i(q^d+1)$, as required by part 2 of Theorem 1.2.

Secondly we claim that for each $1 \leq i \leq n(w_\mathfrak{p}(f), p)$ and for each $1 \leq j \leq i+1$,

$$\partial^j(f_{i-j+1}) = (w_\mathfrak{p}(f) + i)\partial^j(f_{i-j})\partial(g_d) + \partial^{j+1}(f_{i-j})g_d. \tag{3.2}$$

The proof is done by induction on j. For any i in the range and $j = 1$, (3.2) follows by applying ∂ to both sides of the equalities defining the f_i's and remembering that $\partial^2(g_d) = 0$. As an induction step, we suppose that (3.2) is true for $i-1$ and $j-1$, and again by simply applying ∂ we obtain (3.2) for i and j.

Now fix $i = n(w_\mathfrak{p}(f), p)$. Then (3.2) becomes

$$\partial^j(f_{n(w_\mathfrak{p}(f),p)-j+1}) = \partial^{j+1}(f_{n(w_\mathfrak{p}(f),p)-j})g_d \tag{3.3}$$

for $1 \leq j \leq n(w_\mathfrak{p}(f), p)$. Using equation (3.3) recursively we obtain that

$$f_{n(w_\mathfrak{p}(f),p)+1} = \partial^{n(w_\mathfrak{p}(f),p)+1}(f')g_d^{n(w_\mathfrak{p}(f),p)+1}.$$

Since $\Theta^{n(w_\mathfrak{p}(f),p)+1}(f) \equiv f_{n(w_\mathfrak{p}(f),p)+1} \pmod{\mathfrak{p}}$ and $g_d \equiv 1 \pmod{\mathfrak{p}}$, the additional assertion follows. $\square$

4. Three applications

4.1. **Forms of lower filtration than weight.**

Of course, we have the following clear corollary to Theorem 1.3:

Corollary 4.1. *Let f be a Drinfeld modular form in $M_\mathfrak{p}$ for $\mathfrak{p}$ an ideal of A generated by a monic prime polynomial, and assume that f is not identically zero modulo $\mathfrak{p}$. Then*

$$\Theta^i(f) \not\equiv 0 \pmod{\mathfrak{p}} \quad \text{for } 1 \leq i \leq n(w_\mathfrak{p}(f), p).$$

This corollary can be used to detect forms that have lower filtration than weight. For example, consider any Drinfeld modular form over $\mathbb{F}_{25}[T]$ of weight 1376 and an ideal $\mathfrak{p}$ of A generated by a monic prime of degree 2. Suppose further that it can be shown that $\Theta^3(f) \equiv 0 \pmod{\mathfrak{p}}$. Then it must be the case that f has lower filtration modulo $\mathfrak{p}$ than weight, since a form of filtration 1376 would have $\Theta^i(f) \not\equiv 0 \pmod{\mathfrak{p}}$ for $0 \leq i \leq 4$. One can in fact determine that $w_\mathfrak{p}(f) = 128$ in the following manner: As a consequence of Theorem 2.2, we know that the filtration of f must be congruent to 1376 modulo 624. Thus since f has lower filtration than weight, it must have filtration 752 or 128. But if it had filtration 752, the corollary above would say that $\Theta^3(f) \not\equiv 0 \pmod{\mathfrak{p}}$.

4.2. **Vanishing modulo $\mathfrak{p}$ of coefficients.**

One can turn the above idea on its head by constructing forms that have lower filtration than weight and using the theory to deduce the vanishing modulo $\mathfrak{p}$ of some of their coefficients. Consider as a toy example the Drinfeld modular form

$$f = (T^q - T)gh^{q+2} + g^{q+2}h^3 = \sum_{i=3}^{\infty} a_i u^i$$

over $\mathbb{F}_q[T]$. It has weight q^2+4q+1. If one considers a monic prime polynomial of degree greater than or equal to 3, it is clear that this form has filtration equal to its weight. Thus for such primes we have $n(w_{\mathfrak{p}}(f),p)=p-1$, from which we may deduce that there is $i\equiv 1 \pmod p$ such that $a_i\neq 0$, because $\Theta^{p-1}(f)\neq 0$ and Θ^{p-1} annihilates all coefficients but those a_i's that have $i\equiv 1 \pmod p$. However, for an ideal $\mathfrak{p}$ generated by a prime polynomial of degree 2, the form is congruent to gh^3 modulo $\mathfrak{p}$. (By (3.1), $g_2=(T^q-T)h^{q-1}+g^{q+1}$, and $g_2\equiv 1 \pmod{\mathfrak{p}}$ if $\mathfrak{p}$ is generated by a prime polynomial of degree 2.) We will show in Proposition 4.2 that $\Theta^{p-1}(gh^3)=0$, which implies that $\Theta^{p-1}(f)\equiv 0 \pmod{\mathfrak{p}}$. Thus for each $i\equiv 1 \pmod p$, $a_i\equiv 0 \pmod{\mathfrak{p}}$ if $\mathfrak{p}$ is generated by a monic prime polynomial of degree 2.

4.3. **Support of non-zero coefficients of monomials.** As a final application of the theorems collected here, we will show a result on the vanishing of coefficients of monomials.

Proposition 4.2. *Let α and β be non-negative integers and consider the monomial $g^\alpha h^\beta$ which has weight $k=\alpha(q-1)+\beta(q+1)$. Write a for the unique integer such that $0\le a<p$ and $a\equiv\alpha \pmod p$ and similarly write b for the unique integer such that $0\le b<p$ and $b\equiv\beta \pmod p$. Then either $0<b-a<p$ or $b=0$, in which case*

$$\Theta^{n(k,p)+1}(g^\alpha h^\beta)=0 \quad \textit{and} \quad \Theta^i(g^\alpha h^\beta)\neq 0 \quad \textit{for } 1\le i\le n(k,p),$$

or $-p<b-a\le 0$ but $b\neq 0$, in which case

$$\Theta^i(g^\alpha h^\beta)\neq 0 \quad \textit{for } 1\le i<p.$$

Proof. First suppose that $\mathfrak{p}$ is an ideal of A generated by a monic prime polynomial of degree 1. Then

$$g^\alpha h^\beta\equiv h^\beta \pmod{\mathfrak{p}}$$

and so

$$w_{\mathfrak{p}}(g^\alpha h^\beta)=\beta(q+1)\equiv b \pmod p.$$

If $b\neq 0$, then $n(w_{\mathfrak{p}}(g^\alpha h^\beta),p)=p-b$ and by the proof of Theorem 1.3,

$$\Theta^{p-b+1}(g^\alpha h^\beta)\equiv\partial^{p-b+1}(h^\beta)=0 \pmod{\mathfrak{p}}.$$

Finally, if $b=0$, then $n(w_{\mathfrak{p}}(g^\alpha h^\beta),p)=0$, and by Theorem 1.3 we have $\Theta(g^\alpha h^\beta)\equiv\partial(h^\beta)=0 \pmod{\mathfrak{p}}$.

Now suppose that $\mathfrak{p}$ is an ideal of A generated by a monic prime polynomial of degree d, where $d>1$. Then

$$w_{\mathfrak{p}}(g^\alpha h^\beta)=k\equiv b-a \pmod p.$$

We consider two cases:

First suppose that $0<b-a<p$. Then $n(k,p)=p-b+a$, and

$$\Theta^{p-b+a+1}(g^\alpha h^\beta)\equiv\partial^{p-b+a+1}(g^\alpha h^\beta)=0 \pmod{\mathfrak{p}}.$$

But then since $p-b+a+1\ge p-b+1$, $\Theta^{p-b+a+1}(g^\alpha h^\beta)\equiv 0$ modulo every prime ideal in A, and we conclude that $\Theta^{n(k,p)+1}(g^\alpha h^\beta)=0$. We also have that for $1\le i\le p-b+a$, $\Theta^i(g^\alpha h^\beta)\neq 0$ since it is not zero modulo $\mathfrak{p}$ for any ideal generated by a prime of degree greater than 1.

Now suppose that $-p<b-a\le 0$. Then $n(k,p)=a-b$. As above we have that

$$\Theta^i(g^\alpha h^\beta)\neq 0 \quad \text{for } 1\le i\le a-b \tag{4.1}$$

and

$$\Theta^{a-b+1}(g^\alpha h^\beta) \equiv \partial^{a-b+1}(g^\alpha h^\beta) \pmod{\mathfrak{p}}.$$

If $b = 0$, then $\partial^{a+1}(g^\alpha h^\beta) = 0$, and the result follows since $\Theta^{n(k,p)+1}(g^\alpha h^\beta) \equiv 0$ modulo every prime ideal of A. If $b \neq 0$, we have

$$\Theta^{a-b+1}(g^\alpha h^\beta) \equiv \partial^{a-b+1}(g^\alpha h^\beta) \neq 0 \pmod{\mathfrak{p}}. \tag{4.2}$$

If $a = p-1$ and $b = 1$, then $a - b + 1 = p - 1$, and so by combining (4.1) and (4.2) the result follows.

Notice now that since $b \neq 0$ and we have already considered the case $a = p - 1$ and $b = 1$, it only remains to consider the cases where $-p + 3 \leq b - a \leq 0$. In any case we have

$$w_\mathfrak{p}(\partial^{a-b+1}(g^\alpha h^\beta)) = (\alpha - a + b - 1)(q-1) + (\beta + a - b + 1)(q+1) \equiv a - b + 2 \pmod{p}.$$

We now would like to apply Theorem 1.3 to $\partial^{a-b+1}(g^\alpha h^\beta)$. Since $-p+3 \leq b-a \leq 0$, we have $2 \leq a - b + 2 \leq p - 1$. Then

$$n(w_\mathfrak{p}(\partial^{a-b+1}(g^\alpha h^\beta)), p) = p - a + b - 2.$$

Applying Theorem 1.3 to $\partial^{a-b+1}(g^\alpha h^\beta)$, we find that

$$\Theta^{a-b+1+i}(g^\alpha h^\beta) \equiv \Theta^i(\partial^{a-b+1}(g^\alpha h^\beta)) \neq 0 \pmod{\mathfrak{p}}$$

for $1 \leq i \leq p - a + b - 2$ or, combining with (4.1),

$$\Theta^i(g^\alpha h^\beta) \neq 0 \quad \text{for } 1 \leq i \leq p - 1,$$

which is the result we sought. □

References

1. S. Ahlgren and M. Boylan, *Arithmetic properties of the partition function*, Invent. Math. **153** (2003), 487–502. MR2000466 (2004e:11115)
2. V. Bosser and F. Pellarin, *Hyperdifferential properties of Drinfeld quasi-modular forms*, Int. Math. Res. Not. IMRN (2008). MR2428858 (2009e:11092)
3. ———, *On certain families of Drinfeld quasi-modular forms*, J. Number Theory (2009), 2952–2990. MR2560846
4. L. Carlitz, *An analogue of the von Staudt-Clausen theorem*, Duke Math. J. **3** (1937), 503–517. MR1546006
5. ———, *An analogue of the Staudt-Clausen theorem*, Duke Math. J. **7** (1940), 62–67. MR0002995 (2:146e)
6. N. Elkies, K. Ono, and T. Yang, *Reduction of CM elliptic curves and modular function congruences*, Int. Math. Res. Not. IMRN **44** (2005), 2695 – 2707. MR2181309 (2006k:11076)
7. E.-U. Gekeler, *On the coefficients of Drinfeld modular forms*, Invent. Math. **93** (1988), 667–700. MR952287 (89g:11043)
8. L. Gerritzen and M. van der Put, *Schottky groups and Mumford curves*, Lecture Notes in Mathematics, vol. 817, Springer-Verlag, 1980. MR590243 (82j:10053)
9. D. Goss, *π-adic Eisenstein series for function fields*, Compos. Math. **41** (1980), 3–38. MR578049 (82e:10053)
10. J. Lehner, *Further congruence properties of the Fourier coefficients of the modular invariant $j(\tau)$*, Amer. J. Math. **71** (1949), 373–386. MR0027802 (10:357b)
11. K. Ono, *The web of modularity: Arithmetic of the coefficients of modular forms and q-series*, CBMS Regional Conference Series in Mathematics, vol. 102, American Mathematical Society, 2004. MR2020489 (2005c:11053)
12. J.-P. Serre, *Formes modulaires et fonctions zêta p-adiques*, Lecture Notes in Mathematics, vol. 350, pp. 191–268, Springer-Verlag, 1973. MR0404145 (53:7949a)

13. H.P.F. Swinnerton-Dyer, *On ℓ-adic representations and congruences for coefficients of modular forms*, Lecture Notes in Mathematics, vol. 350, pp. 1–55, Springer-Verlag, 1973. MR0406931 (53:10717a)
14. Y. Uchino and T. Satoh, *Function field modular forms and higher derivations*, Math. Ann. **311** (1998), 439–466. MR1637907 (99j:11044)

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