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Some Results on Two Conjectures of Schützenberger

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Abstract. We present some partial results concerning two conjectures of Schützenberger on evacuations of Young tableaux.

1 Introduction

In [8] M.-P. Schützenberger defined a construction on Young tableaux, the so-called *evacuation* of a tableau, which associates to any Young tableau T a new one, namely $ev(T)$, by means of successive applications of slides of cells, (see also [2, 9, 11]). This algorithm has remarkable properties in relation to the Robinson–Schensted–Knuth correspondence (see [10]). In this paper we consider two conjectures made by Schützenberger, one of which was left unpublished.

Conjecture 1 ([10]) *Let T and T' be two Young tableaux that differ only by a transposition of consecutive integers. Then the evacuations of T and T' will differ by a cycle of even length.*

Conjecture 2 *Let T and T' be two Young tableaux of rectangular shape that differ only by a transposition of consecutive integers. Then if we iterate the full promotion k times, the corresponding tableaux will differ by a hook cycle for any k.*

After presenting some basic terminology and notation, we present a counterexample to Conjecture 2 as stated. A plausible alternative is suggested (Conjecture 3). We offer some counterexamples to various possible strengthenings of Conjecture 1. We then prove that Conjecture 1 holds in various special cases. In Theorem 3.1, we prove that it holds in the case of rectangular tableaux; in Theorem 3.2 we prove that if the evacuations differ by a cycle, then it must be of even length, and in Corollary 3.5 we verify the conjecture when the transposition exchanges "small" integers. We shall also prove that the conjecture holds when the transposition is $(n-1 \quad n)$, a result already known to Foata [1], but we present a simpler and more explicit proof (Theorem 3.9). Furthermore we prove that the evacuations actually differ by a hook cycle

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in this case. Finally, we show that after one full promotion of tableaux that differ by a transposition of consecutive integers, the resulting tableaux differ by a hook cycle (Theorem 4.1).

2 Preliminaries

We refer the reader to Sagan's monograph [7] for all basic facts concerning tableaux. As usual S_n shall denote the symmetric group on $[n] = \{1, \dots, n\}$.

A *partition* λ of k parts of a positive integer n is a decreasing sequence $\lambda = (\lambda_1, \dots, \lambda_k)$ of positive integers such that $\sum_{i=1}^{k} \lambda_i = n$. To say that λ is a partition of n we will use one of the following notations: $|\lambda| = n$ or $\lambda \vdash n$. The *Ferrers diagram* or *shape* of a partition λ is an array of n cells (or boxes) into k left justified rows, where row i contains λ_i boxes. More precisely, the diagram is the subset of $\mathbb{N} \times \mathbb{N}$ defined by

$$\text{diag}(\lambda) := \{(i, j) \in \mathbb{N} \times \mathbb{N} : 1 \leq i \leq k, 1 \leq j \leq \lambda_i\}.$$

A *standard Young tableau* of shape λ is a filling of the Ferrers diagram of λ with entries in $[n]$ in such a way that the entries are strictly increasing from left to right on the rows and from bottom up on the columns.

In what follows we will deal exclusively with standard Young tableaux, and hence *tableau* will always be understood to mean *standard Young tableau.* Let T be a tableau with n entries. We define the *trace* of T, denoted by $tr(T)$, as the following sequence of cells: start from the cell containing 1 and choose the neighboring cell containing the smallest value as the next cell. For each $1 \leq k \leq n$ we define the *promotion* ∂_k as the following operation, a map which sends the tableaux T of shape λ to another tableau of the same shape.

(i) Delete 1 from T.
(ii) For every element less than or equal to k in $tr(T)$, slide the value onto the preceding cell in the trace.
(iii) Subtract 1 from every element in the tableau less than or equal to k.
(iv) Place k in the empty cell created by sliding the trace.

For a tableau of size n, we call $\partial_n(T)$ the *full promotion* of T. We define the *evacuation* of a tableau T as $\partial_1\partial_2 \dots \partial_{n-1}\partial_n(T)$ and denote it by $ev(T)$.

We shall also use the following notation. If T is a tableau of size n and σ is a permutation in S_n, then σT shall denote the tableau obtained from T by replacing each entry j by $\sigma(j)$. If $S = \sigma T$ we say that tableaux S and T *differ by* σ. We shall also use the following operators on Young tableaux, introduced by Haiman in [3]: for every $1 \leq i \leq n-1$, let s_i denote the transposition $(i \quad i+1)$ and define

$$r_i(T) = \begin{cases} s_i T, & \text{if } s_i T \text{ is a standard Young tableau;} \\ T, & \text{otherwise.} \end{cases}$$

For example,

$$T = \begin{array}{|c|c|c|} \hline 6 & \multicolumn{2}{c}{} \\ \cline{1-1} 4 & 5 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 3 \\ \hline \end{array}, \quad r_5 T = \begin{array}{|c|c|c|} \hline 5 & \multicolumn{2}{c}{} \\ \cline{1-1} 4 & 6 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 3 \\ \hline \end{array}, \quad r_2 T = \begin{array}{|c|c|c|} \hline 6 & \multicolumn{2}{c}{} \\ \cline{1-1} 4 & 5 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 3 \\ \hline \end{array}.$$

The following result is from [3] and [5].

Lemma 2.1 *Let T be a tableau of size n. For any $1 \leq k \leq n$ we have $\partial_k(T) = r_{k-1} \cdots r_2 r_1(T)$.*

A permutation is a *hook cycle* if, in cycle notation, it is of the form $(a_1 a_2 \cdots a_t)$, with $a_1 < a_2 < \cdots < a_j > a_{j+1} > \cdots > a_t$ for some $j \leq t$, where a_1 is the smallest element of the cycle.

We shall also require some results of Knuth and Schützenberger; the reader can refer to [7] for the proofs that are omitted here.

The Robinson–Schensted–Knuth (RSK) correspondence establishes a bijection between pairs (P, Q) of standard Young tableaux of the same shape and of size n and the permutations of S_n. Given a permutation π, we denote the associated tableaux $P(\pi)$ and $Q(\pi)$ respectively; P is called the *insertion tableau*, and Q is the *recording tableau.*

It will often be convenient to use the two-line notation for permutations:

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix}$$

denotes the permutation that satisfies $\pi(i) = b_i$ for all $1 \leq i \leq n$.

We define the *row word* of a tableau, word(T), to be the permutation obtained from reading the entries of the tableau from left to right, starting from the top row, and inserting these entries in the second line of the two-line notation of π. We observe that $P(\text{word}(T)) = T$.

For example, if

$$T = \begin{array}{|c|c|c|c} \hline 10 \\ \hline 6 & 8 & 11 \\ \hline 3 & 4 & 7 \\ \hline 1 & 2 & 5 & \multicolumn{1}{c|}{9} \\ \hline \end{array}\ ,$$

then

$$\text{word}(T) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 10 & 6 & 8 & 11 & 3 & 4 & 7 & 1 & 2 & 5 & 9 \end{pmatrix}.$$

Lemma 2.2 *If $\pi \in S_n$, then $P(\pi^{-1}) = Q(\pi)$ and $Q(\pi^{-1}) = P(\pi)$.*

If $\pi = \left(\begin{smallmatrix} 1 & \cdots & n \\ x_1 & \cdots & x_n \end{smallmatrix}\right)$, we define the *reversal* of π by $\pi^r = \left(\begin{smallmatrix} 1 & \cdots & n \\ x_n & \cdots & x_1 \end{smallmatrix}\right)$ and call $w_0 = \left(\begin{smallmatrix} 1 & \cdots & n \\ n & \cdots & 1 \end{smallmatrix}\right)$ the longest permutation (reversal of the identity permutation). The *transpose* of a tableau T, denoted by T^t, is the tableau satisfying the following: its value in cell (j, i) is the value of T at cell (i, j).

Lemma 2.3 *If $\pi \in S_n$, then $P(\pi^r) = P(\pi)^t$ and $Q(\pi^r) = ev(Q(\pi))^t$.*

From these results it follows that for any π we have

$$ev(P(\pi)) = P((((\pi^{-1})^r)^{-1})^r). \tag{2.1}$$

As it is easy to go from a tableau T to its row word π and then to calculate $\sigma = (((\pi^{-1})^r)^{-1})^r$, we only have to produce $P(\sigma)$ to obtain the evacuated tableau. We use Knuth's algorithm, as presented in [4] to do so (see Section 5.1.4 "Algorithm S"). First, we build a table using the following rules:

(i) If $\sigma = \left(\begin{smallmatrix} 1 & \cdots & n \\ x_1 & \cdots & x_n \end{smallmatrix}\right)$, then line 1 consists of the entries from x_1 to x_n.
(ii) For $k \geq 1$, the line $k+1$ is created from line k in the following fashion.
   (a) Let $p \leftarrow \infty$.
   (b) Let the column j be the leftmost column such that the line k contains an integer strictly less than p and the line $k+1$ is empty. If this column does not exist, and if $p = \infty$, the line $k+1$ is complete; if it does not exist and $p < \infty$, return to step (a).
   (c) Insert p in column j, line $k+1$; let p have the value found on column j of line k and return to step (b).

To construct $P(\sigma)$ from this table, it suffices to let the elements of row k of $P(\sigma)$ be those present in line k of the table but not in line $k+1$.

3 Results

3.1 Rectangular Tableaux

We now proceed to prove Conjecture 1 in the case where the tableaux have rectangular shape.

Theorem 3.1 *Let T and T' be two Young tableaux of rectangular shape differing only by the transposition $(i \quad i+1)$. Then $ev(T)$ and $ev(T')$ differ by the transposition $(n-i \quad n+1-i)$.*

Proof Let $T = \boxed{a_{i,j}}$, $1 \leq i \leq l$, $1 \leq j \leq m$, $n = l \cdot m$. Then

$$\text{word}(T) = \begin{pmatrix} 1 & \cdots & m & m+1 & \cdots & n-m+1 & \cdots & n \\ a_{l,1} & \cdots & a_{l,m} & a_{l-1,1} & \cdots & a_{1,1} & \cdots & a_{1,m} \end{pmatrix}$$

so

$$((\text{word}(T)^{-1})^r)^{-1} =$$

$$\begin{pmatrix} 1 & \ldots & m & m+1 & \cdots & n \\ n+1-a_{l,1} & \cdots & n+1-a_{l,m} & n+1-a_{l-1,1} & \cdots & n+1-a_{1,m} \end{pmatrix}.$$

During the construction of $ev(T)^t$ using Knuth's algorithm, when creating the table's second line, we observe that since T is a standard Young tableau, we have for all $1 \leq i \leq m$ that

$$n+1-a_{i,1} > \cdots > n+1-a_{i,j} > \cdots > n+1-a_{i,m}$$

and $n+1-a_{i,m} < n+1-a_{k,j}$ for all $1 \leq k \leq i, 1 \leq j \leq m$. Hence the first row of $ev(T)^t$ will have as entries $\{n+1-a_{i,m}\}$, for $1 \leq i \leq l$. Since these entries do not appear on the table's second line, it is easy to see that the second row of $ev(T)^t$ will contain the entries $\{n+1-a_{i,m-1}\}$, $1 \leq i \leq l$, since for all $1 \leq i \leq m-1$ we have $n+1-a_{i,1} < \cdots < n+1-a_{i,m-1}$ and $n+1-a_{i,m-1} < n+1-a_{k,j}$ for all $1 \leq k \leq i, 1 \leq j \leq m-1$. In general, row k of $ev(T)^t$ will contain the entries $\{n+1-a_{i,m+1-k}\}$, $1 \leq i \leq l$.

We therefore have $ev(T)^t = \boxed{n+1-a_{l+1-i,m+1-j}}$.

It is now clear that if T and T' are two standard Young tableaux of rectangular shape differing only by the transposition $(i \quad i+1)$, their evacuated tableaux will differ by the transposition $(n-i \quad n+1-i)$. ■

3.2 Cycles of Even Length

We now show that if two tableaux T and T' differ by a transposition of consecutive integers, and $ev(T)$ and $ev(T')$ differ by a cycle, then this cycle must be of even length. We say that a pair of entries (i, j) of a tableau T is an *inversion* if $i < j$ and i lies on a row strictly below the row of j in the tableau; let $\mathrm{inv}(T)$ denote the number of inversions in T. We define the sign of a tableau T as follows: $\mathrm{sign}(T) = (-1)^{\mathrm{inv}(T)}$.

Theorem 3.2 *Let T and T' be two standard Young tableaux that differ by a transposition of consecutive integers. If $ev(T)$ and $ev(T')$ differ by a cycle, then this cycle is of even length.*

Proof Let T and T' be two tableaux differing by a transposition of consecutive integers. Let π be the permutation whose insertion and recording tableaux are T and T' respectively, so that $P = P(\pi) = T$ and $Q = Q(\pi) = T'$.

Then by Lemmas 2.2 and 2.3 and equation (2.1) we have that the permutation $(((\pi^{-1})^r)^{-1})^r$ has $ev(P)$ and $ev(Q)$ as insertion and recording tableaux respectively. It is easy to see that

$$(((\pi^{-1})^r)^{-1})^r = \begin{pmatrix} 1 & \cdots & n-1 & n \\ n+1-\pi(n) & \cdots & n+1-\pi(2) & n+1-\pi(1) \end{pmatrix}$$

or equivalently $(((\pi^{-1})^r)^{-1})^r = w_0 \pi w_0$ where

$$w_0 = \begin{pmatrix} 1 & \cdots & n-1 & n \\ n & \cdots & 2 & 1 \end{pmatrix}.$$

In [6, Theorem 4.3], it is shown that for any permutation π with associated tableaux P and Q we have that

$$\mathrm{sign}(\pi) = \mathrm{sign}(P) * \mathrm{sign}(Q) * (-1)^e \tag{3.1}$$

where e is the total length of all even-indexed rows of P (or Q). Since P, Q, $ev(P)$, and $ev(Q)$ all have the same shape and multiplying π on both sides by w_0 does not change the parity, we obtain from equation (3.1) applied to π and $(((\pi^{-1})^r)^{-1})^r$ that

$$\begin{aligned} \mathrm{sign}(P) * \mathrm{sign}(Q) &= \mathrm{sign}(\pi) * (-1)^e \\ &= \mathrm{sign}(w_0 \pi w_0) * (-1)^e \\ &= \mathrm{sign}(ev(P)) * \mathrm{sign}(ev(Q)). \end{aligned}$$

By hypothesis, the difference in the number of inversions in $P = T$ and $Q = T'$ is odd. It follows that the same must hold for $ev(P)$ and $ev(Q)$, hence the two evacuated tableaux differ by the product of an odd number of transpositions. ■

3.3 Special Transpositions

We now prove Conjecture 1 for specific transpositions.

Let α and β be permutations in S_n. We say that α and β differ by a *Knuth relation* if for some i there exist x, y, z with $x < y < z$ such that we have one of the following two cases:

$$\alpha = \begin{pmatrix} \cdots & i & i+1 & i+2 & \cdots \\ \cdots & y & x & z & \cdots \end{pmatrix} \text{ and } \beta = \begin{pmatrix} \cdots & i & i+1 & i+2 & \cdots \\ \cdots & y & z & x & \cdots \end{pmatrix}$$

or

$$\alpha = \begin{pmatrix} \cdots & i & i+1 & i+2 & \cdots \\ \cdots & x & z & y & \cdots \end{pmatrix} \text{ and } \beta = \begin{pmatrix} \cdots & i & i+1 & i+2 & \cdots \\ \cdots & z & x & y & \cdots \end{pmatrix}$$

It is immediate to verify that α^{-1} and β^{-1} differ by a Knuth relation if and only if, for some $x < y < z$ there exists some k such that one of the two following cases occurs:

$$\alpha = \begin{pmatrix} \cdots & x & \cdots & y & \cdots & z & \cdots \\ \cdots & k+2 & \cdots & k & \cdots & k+1 & \cdots \end{pmatrix} \text{ and}$$

$$\beta = \begin{pmatrix} \cdots & x & \cdots & y & \cdots & z & \cdots \\ \cdots & k+1 & \cdots & k & \cdots & k+2 & \cdots \end{pmatrix}$$

or

$$\alpha = \begin{pmatrix} \cdots & x & \cdots & y & \cdots & z & \cdots \\ \cdots & k & \cdots & k+2 & \cdots & k+1 & \cdots \end{pmatrix} \text{ and}$$

$$\beta = \begin{pmatrix} \cdots & x & \cdots & y & \cdots & z & \cdots \\ \cdots & k+1 & \cdots & k+2 & \cdots & k & \cdots \end{pmatrix}.$$

If this occurs we say that α and β differ by a *dual Knuth relation*.

The next lemma follows immediately from the proof of Lemma 4.1 in [6] and Lemma 2.2.

Lemma 3.3 *Let π and σ be two permutations differing by a dual Knuth relation. Then $P(\sigma)$ can be obtained from $P(\pi)$ by exchanging two consecutive entries. More precisely, if π and σ differ by a dual Knuth relation applied on the triple $\{i, i+1, i+2\}$, then $P(\pi)$ and $P(\sigma)$ differ by the position of either i and $i+1$ or $i+1$ and $i+2$.*

We can now prove the following.

Lemma 3.4 *Let T and T' be two tableaux whose row words differ by a dual Knuth relation. Then $ev(T)$ and $ev(T')$ differ by a transposition of consecutive integers.*

Proof Let α and β be the row words of T and T' respectively, and suppose that they differ by a dual Knuth relation. Then it is clear that the permutations $\alpha' = (((\alpha^{-1})^r)^{-1})^r$ and $\beta' = (((\beta^{-1})^r)^{-1})^r$ also differ by a dual Knuth relation. Since we have $P(\alpha') = ev(T)$ and $P(\beta') = ev(T')$ by equation (2.1), we are done by Lemma 3.3. ∎

Corollary 3.5 *Let T and T' be two Young tableaux differing only by the transposition $(2\,3)$ or $(3\,4)$. Then $ev(T)$ and $ev(T')$ differ by a transposition of consecutive integers.*

Proof It suffices to prove that if T and T' differ only by the transposition $(2\,3)$ or $(3\,4)$ then they satisfy the conditions of Lemma 3.4. If the two tableaux differ by $(2\,3)$, the only possible configuration when restricting to the entries $1, 2, 3$ is

$$T = \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array} \quad \text{and } T' = \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} .$$

In this case

$$\text{word}(T) = \begin{pmatrix} \cdots & x & \cdots & y & z & \cdots \\ \cdots & 3 & \cdots & 1 & 2 & \cdots \end{pmatrix} \text{ and}$$
$$\text{word}(T') = \begin{pmatrix} \cdots & x & \cdots & y & z & \cdots \\ \cdots & 2 & \cdots & 1 & 3 & \cdots \end{pmatrix}$$

and therefore word(T) and word(T') differ by a dual Knuth relation.

Up to transposition, the only configuration for $(3\,4)$ where 3 and 4 can be swapped is

$$T = \begin{array}{|c|cc}\hline 4 & & \\ \hline 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} \\ \hline \end{array} \quad \text{and } T' = \begin{array}{|c|cc}\hline 3 & & \\ \hline 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{4} \\ \hline \end{array} .$$

Once again, word(T) and word(T') differ by a dual Knuth relation. ■

For our next results, we shall use the description given in Lemma 2.1 of $ev(T)$ as a sequence of successive promotions. We shall first need a few auxiliary lemmas. Given a permutation α of the set $[n]$, let $Fix(\alpha) := \{i \in [n] : \alpha(i) = i\}$ be the subset of $[n]$ consisting of the points fixed by α. The *support* of α is the subset $\text{Supp}(\alpha)$ of $[n]$ complementary to the fixed points subset of α, that is $\text{Supp}(\alpha) := \{i \in [n] : \alpha(i) \neq i\}$.

Lemma 3.6 *Let $k \geq 1$. Let S and T be two Young tableaux of size n of the same shape λ, and suppose that S and T differ by a hook cycle permutation α with support contained in $\{n-k+1, \ldots, n\}$. Then $r_{n-k}S$ and $r_{n-k}T$ differ by a hook cycle γ with support contained in $\{n-k, \ldots, n\}$.*

Proof We want to compare $r_{n-k}S$ and $r_{n-k}T$ where S and T differ by α.

Case 1 If r_{n-k} acts in the same way on both tableaux S and T, then we have two subcases:

(i) If r_{n-k} acts as the identity on both S and T, then $\gamma = \alpha$, a hook cycle;

(ii) If $r_{n-k} = (n-k \quad n-k+1)$ acts as the adjacent transposition s_{n-k} on both S and T, then $\gamma = (n-k \quad n-k+1) \circ \alpha \circ (n-k \quad n-k+1)$ is the conjugate of α by s_{n-k}. Since $n-k$ does not appear in α, we get $\gamma = \alpha$ when $n-k+1$ does not appear in α (hence α is a hook cycle) or $\gamma = \alpha'$, where α' is the cycle obtained by α by replacing $n-k+1$ by $n-k$. Since $n-k+1$ was the smallest element of the hook cycle α, α' is also a hook cycle.

Case 2 If r_{n-k} does not act in the same way on both tableaux S and T, (*i.e.*, acts on one side as the identity, on the other tableau as s_{n-k}), then $n-k+1$ must appear in α, and $\gamma = (n-k \quad n-k+1) \circ \alpha$ or $\gamma = \alpha \circ (n-k \quad n-k+1)$. Either way, γ is a cycle obtained from α by inserting $n-k$ next to $n-k+1$ (on one side or the other). Since $n-k+1$ was the smallest element in α, γ is still a hook cycle. ■

Lemma 3.7 *Let $j \geq 1$. Let S and T be two Young tableaux of size n of the same shape λ, and suppose that S and T differ by a hook cycle permutation α with support contained in $\{1, \ldots, j\}$. Then r_jS and r_jT differ by a hook cycle γ with support contained in $\{1, \ldots, j+1\}$.*

Proof The proof is completely analogous to the proof of Lemma 3.6. ■

Lemma 3.8 *Suppose that in a tableau T the entries i and $i+1$ are neither in the same row nor in the same column (*i.e., *$r_i(T) = s_i(T)$). Then one of the following holds:*

(a) $r_{i-1}r_ir_{i-1}(T) = r_ir_{i-1}r_i(T)$.
(b) $r_{i-1}r_ir_{i-1}(T) = r_i(T)$ *and* $r_ir_{i-1}r_i(T) = id(T)$.

Proof The proof consists of a simple case-by-case analysis of the relative positions of entries $i-1, i$ and $i+1$. It is easy to see that, up to transposes of tableaux and reordering of the three values, there are only 4 cases to consider:

(i) *All three entries are in different rows and columns.* It is then obvious that case (a) holds.
(ii) *$i-1$ and i are in the same row; $i-1$ and $i+1$ are in the same column.* It is easy to see that this is case (b).
(iii) *$i-1$ and i are in the same row; $i-1$ and $i+1$ are not in the same column.* An easy computation shows that this is case (a).
(iv) *$i-1$ and $i+1$ are in the same row, and i is not in the same column as $i-1$ or $i+1$.* Once again a simple computation shows that (a) holds. ■

Theorem 3.9 *Let S and T be two Young tableaux differing by $s_{n-1} = (n-1 \quad n)$. Then $ev(S)$ and $ev(T)$ differ by a hook cycle of even length.*

Proof By Theorem 3.2 it suffices to prove the evacuated tableaux differ by a hook cycle. Let (S, T) be a pair of Young tableaux which constitute a counterexample of minimal size, that is, $|S| = |T| = n$ is minimal among all pairs of tableaux which are counterexamples of the statement. Apply the first promotion ∂_n to S and T. By Lemma 2.1 and since r_i and r_j commute when $|i-j| \neq 1$, we have that

$$\partial_n(S) = \partial_n(r_{n-1}(T)) = r_{n-1}\cdots r_1(r_{n-1}(T)) = r_{n-1}r_{n-2}r_{n-1}r_{n-3}\cdots r_1(T).$$

Applying Lemma 3.8 to tableau $W = r_{n-3}\cdots r_1(T)$, only two cases occur.

Case 1 We have that $r_{n-1}r_{n-2}r_{n-1} = r_{n-2}r_{n-1}r_{n-2}$ and thus $\partial_n(r_{n-1}(T)) = r_{n-2}\partial_n(T)$; in other words, the tableaux $\partial_n(S)$ and $\partial_n(T)$ differ only by the adjacent transposition $(n-2 \quad n-1)$. This is impossible. Otherwise consider the tableaux S' and T' of size $n-1$, obtained from $\partial_n(S)$ and $\partial_n(T)$ by removing the cell containing n. They differ by $(n-2 \quad n-1)$, and the successive application of the promotions

$\partial_{n-1}, \ldots, \partial_1$ will yield $ev(S')$ and $ev(T')$, which are exactly the tableaux obtained from $ev(S)$ and $ev(T)$ by removing the cell containing n. Hence S' and T' would be a counterexample of smaller size.

Case 2 Otherwise $r_{n-2}r_{n-1}r_{n-2} = r_{n-1}$ and we obtain

$$\begin{aligned} r_{n-1}\partial_n(T) &= r_{n-1}r_{n-2}r_{n-2}\partial_n(T) = r_{n-1}r_{n-2}r_{n-2}r_{n-1}r_{n-2}W \\ &= r_{n-1}r_{n-2}r_{n-1}W = \partial_n(S). \end{aligned}$$

Hence the tableaux $\partial_n(S)$ and $\partial_n(T)$ differ only by the adjacent transposition $(n-1\; n)$. In this case, simply apply Lemma 3.6 to $\partial_n(S)$ and $\partial_n(T)$ with $k = 2$ and $\alpha = (n-1\; n)$; then $\partial_{n-1}\partial_n(S) = r_{k-2}\cdots r_1\partial_n(S)$ and $\partial_{n-1}\partial_n(T) = r_{k-2}\cdots r_1\partial_n(T)$ will differ by a hook cycle permutation with support contained in $\{n-2, n-1, n\}$. Inductively, we can again apply the same lemma for the successive applications of promotions, up to $ev(S)$ and $ev(T)$, thus proving the claim. ■

4 Some Counterexamples

4.1 Counterexamples to Conjecture 2

It turns out that Conjecture 2 is false, as the following counterexample shows.

If we take T to be the tableau below and $T' = (8\;9)T$, after 4 full promotions, the tableaux will differ by (5 6 9 8 10 7), which is not a hook cycle.

$T =$

7	10	13	16
4	9	12	15
3	5	8	14
1	2	6	11

and $T' =$

7	10	13	16
4	8	12	15
3	5	9	14
1	2	6	11

$\partial_{16}^4 T =$

6	12	13	16
5	9	11	15
3	4	8	14
1	2	7	10

and $\partial_{16}^4 T' =$

9	12	13	16
6	8	11	15
3	4	10	14
1	2	5	7

However, we still cannot find a counterexample to the following weaker version of Conjecture 2.

Conjecture 3 *Let T and T' be two tableaux of rectangular shape which differ only by a transposition of consecutive integers. Then if we iterate the full promotion k times, the corresponding tableaux will differ by a cycle of even length, for any k.*

4.2 A Result with Full Promotion and Hook Cycles

While Conjecture 2 is false, when applying the full promotion more than once to rectangular shapes, we present here another partial result, this time valid for general tableaux and for all transpositions of consecutive integers, but holding only for one full promotion of the tableaux.

Theorem 4.1 *Let S and T differ by the adjacent transposition* $(i\ i+1)$*. Then* $\partial_n(S)$ *and* $\partial_n(T)$ *will differ by a hook cycle permutation.*

Proof First observe that, if we let $U = r_{i-2} \cdots r_1(T)$, then

$$\begin{aligned}\partial_n(S) &= \partial_n(r_i(T)) \\ &= r_{n-1} \cdots r_1(r_i(T)) \\ &= r_{n-1} \cdots r_i r_{i-1} r_i (r_{i-2} \cdots r_1(T)) \\ &= r_{n-1} \cdots r_i r_{i-1} r_i(U)\end{aligned}$$

and that

$$\partial_n(T) = r_{n-1} \cdots r_1(T) = r_{n-1} \cdots r_i r_{i-1}(r_{i-2} \cdots r_1(T)) = r_{n-1} \cdots r_i r_{i-1}(U).$$

Clearly we may apply Lemma 3.8 to U.

(i) Suppose first that $r_{i-1} r_i r_{i-1}(U) = r_i r_{i-1} r_i(U)$. Then we obtain (since r_{i-1} commutes with r_j for all $j \geq i+1$) that

$$\begin{aligned}\partial_n(S) &= r_{n-1} \cdots r_{i+1} r_i r_{i-1} r_i(U) \\ &= r_{n-1} \cdots r_{i+1} r_{i-1} r_i r_{i-1}(U) \\ &= r_{i-1} r_{n-1} \cdots r_{i+1} r_i r_{i-1}(U) \\ &= r_{i-1} \partial_n(T).\end{aligned}$$

(ii) Otherwise we have that $r_i r_{i-1} r_i(U) = id(U)$ thus

$$\partial_n(S) = r_{n-1} \cdots r_{i+1} r_i r_{i-1} r_i(U) = r_{n-1} \cdots r_{i+1}(U)$$

and

$$\partial_n(T) = r_{n-1} \cdots r_{i+1}(r_i r_{i-1}(U))$$

where obviously U and $r_i r_{i-1}(U)$ differ by a hook cycle with support in $\{1, \ldots, i+1\}$. Invoking Lemma 3.7 recursively, we obtain the desired result. ∎

4.3 More about Conjecture 1

As far as Conjecture 1 is concerned, we have verified it by computer for tableaux of all shapes up to size 15. Comparing both conjectures, it is tempting to try to strengthen Conjecture 1 in various ways to obtain a more manageable problem, but it turns out that all natural variants of the conjecture are false. First, we show that the hypothesis that the original transposition exchanges consecutive integers is essential. For instance, consider the following statement, which is a variation on Conjecture 1.

> *Let* T *and* T' *be two standard Young tableaux differing by a transposition of any two integers (not necessarily consecutive). Then* $ev(T)$ *and* $ev(T')$ *will differ by a cycle of even length.*

This is false. If T is the tableau below, and $T' = (6\,8)T$, then their evacuations will differ by the permutation $(3\,5)(4\,6\,7)$:

$$T = \begin{array}{lll} 9 & & \\ 6 & & \\ 4 & & \\ 3 & & \\ 2 & 7 & \\ 1 & 5 & 8 \end{array} \quad \text{and } T' = \begin{array}{lll} 9 & & \\ 8 & & \\ 4 & & \\ 3 & & \\ 2 & 7 & \\ 1 & 5 & 6 \end{array}$$

$$ev(T) = \begin{array}{lll} 9 & & \\ 8 & & \\ 7 & & \\ 5 & & \\ 2 & 6 & \\ 1 & 3 & 4 \end{array} \quad \text{and } ev(T') = \begin{array}{lll} 9 & & \\ 8 & & \\ 4 & & \\ 3 & & \\ 2 & 7 & \\ 1 & 5 & 6 \end{array} .$$

Next variation: we might try to strengthen the conclusion to obtain a hook cycle:

Let T and T' be two tableaux which differ only by a transposition of consecutive integers. Then the evacuations of T and T' will differ by a hook cycle.

This is false too. If we take T to be the tableau below and $T' = (7\ 8)T$, then their evacuations differ by the permutation $(5\ 7\ 10\ 8\ 9\ 6)$, which is not a hook cycle.

$$T = \begin{array}{llll} 10 & & & \\ 6 & 8 & 11 & \\ 3 & 4 & 7 & \\ 1 & 2 & 5 & 9 \end{array} \quad \text{and } T' = \begin{array}{llll} 10 & & & \\ 6 & 7 & & \\ 3 & 4 & 8 & \\ 1 & 2 & 5 & 9 \end{array}$$

$$ev(T) = \begin{array}{llll} 7 & & & \\ 5 & 10 & 11 & \\ 3 & 4 & 8 & \\ 1 & 2 & 6 & 9 \end{array} \quad \text{and } ev(T') = \begin{array}{llll} 10 & & & \\ 7 & 8 & 11 & \\ 3 & 4 & 9 & \\ 1 & 2 & 5 & 6 \end{array} .$$

A third variation: instead of hook cycles, we might ask that the tableaux differ by a cycle of even length, but this after each successive promotion.

Let T and T' be two tableaux which differ only by a transposition of consecutive integers. During their respective evacuations, the tableaux will differ by a cycle of even length.

This in fact is still false even if we assume that the tableaux have rectangular shape. If T is the tableau below and T' differs from T by $(7\,8)$, then after 7 promotions, the resulting tableaux differ by a cycle of odd length:

$$T = \begin{array}{llll} 7 & 9 & 13 & 16 \\ 4 & 6 & 12 & 15 \\ 2 & 5 & 10 & 14 \\ 1 & 3 & 8 & 11 \end{array} \quad \text{and } T' = \begin{array}{llll} 8 & 9 & 13 & 16 \\ 4 & 6 & 12 & 15 \\ 2 & 5 & 10 & 14 \\ 1 & 3 & 7 & 11 \end{array}$$

$$\partial_{10}\partial_{11}\partial_{12}\partial_{13}\partial_{14}\partial_{15}\partial_{16}T = \begin{array}{|c|c|c|c|}\hline 6&9&14&16\\\hline 5&8&12&15\\\hline 2&7&11&13\\\hline 1&3&4&10\\\hline\end{array} \quad \text{and } \partial_{10}\partial_{11}\partial_{12}\partial_{13}\partial_{14}\partial_{15}\partial_{16}T = \begin{array}{|c|c|c|c|}\hline 9&10&14&16\\\hline 6&8&12&15\\\hline 5&7&11&13\\\hline 1&2&3&4\\\hline\end{array}.$$

These differ by the cycle (2 5 6 9 10 4 3).

Another possible modification would be to ask that the tableaux differ by a hook cycle after each successive promotion, instead of a cycle of even length.

Let T and T' be two tableaux of rectangular shape differing only by a transposition of consecutive integers. During their respective evacuations, the tableaux will always differ by a hook cycle.

False again: if T is the tableau below, and $T' = (7\,8)T$, then after 3 promotions the resulting tableaux differ by the cycle (5 6 9 8 10 7):

$$T = \begin{array}{|c|c|c|c|}\hline 9&12&15&16\\\hline 6&8&11&14\\\hline 2&4&7&13\\\hline 1&3&5&10\\\hline\end{array} \quad \text{and } T' = \begin{array}{|c|c|c|c|}\hline 9&12&15&16\\\hline 6&7&11&14\\\hline 2&4&8&13\\\hline 1&3&5&10\\\hline\end{array}$$

$$\partial_{14}\partial_{15}\partial_{16}T = \begin{array}{|c|c|c|c|}\hline 6&12&14&16\\\hline 5&9&11&15\\\hline 3&4&8&13\\\hline 1&2&7&10\\\hline\end{array} \quad \text{and } \partial_{14}\partial_{15}\partial_{16}T = \begin{array}{|c|c|c|c|}\hline 9&12&14&16\\\hline 6&8&11&15\\\hline 3&4&10&13\\\hline 1&2&5&7\\\hline\end{array}.$$

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