Drinfeld modular forms modulo $\mathfrak{p}$ and Weierstrass points on Drinfeld modular curves

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Abstract

We consider the Drinfeld setting which offers analogues for function fields of some aspects of the theory of modular forms, modular curves and elliptic curves. In this setting the field $\mathbb{F}_q(T)$, for q a power of a prime, plays the role of $\mathbb{Q}$, and all objects are defined over a complete, algebraically closed field of positive characteristic containing $\mathbb{F}_q(T)$.

We first consider the action of the Hasse derivatives on Drinfeld modular forms, which were shown by Uchino and Satoh to act as differential operators on the algebra of Drinfeld quasi-modular forms. While these operators do not preserve modularity, we show that they do preserve modularity modulo $\mathfrak{p}$ for $\mathfrak{p}$ a prime ideal of $\mathbb{F}_q[T]$. We also study the behavior of the filtration under the action of the first Hasse derivative, and obtain results analogous to those obtained by Serre and Swinnerton-Dyer about Ramanujan's Θ-operator in the classical setting.

We then consider the family of modular curves $X_0(\mathfrak{p})$ constructed by Drinfeld, and we study their Weierstrass points, a finite set of points of geometric interest. These curves are moduli spaces for Drinfeld modules with level structure, which are the objects which in our setting play a role analogous to that of elliptic curves. Previous work of Baker shows that for each Weierstrass point, the reduction modulo $\mathfrak{p}$ of the underlying Drinfeld module is supersingular. We study a modular form W for $\Gamma_0(\mathfrak{p})$ whose divisor is closely related to the set of Weierstrass points, an idea first presented by Rohrlich in the classical setting. To this end, we first establish a one-to-one correspondence between certain Drinfeld modular forms on $\Gamma_0(\mathfrak{p})$ and forms on the full modular group. In certain cases we can then use knowledge about the action of the Hasse derivatives to compute

explicitly a form $\widetilde{W}$ that is congruent to W modulo $\mathfrak{p}$. This allows us to obtain an analogue of Rohrlich's result, which is the first important step towards obtaining a more precise relationship between the supersingular locus and Weierstrass points on $X_0(\mathfrak{p})$, as illustrated by Ahlgren and Ono in the classical setting.

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Notations

$A = \mathbb{F}_q[T]$	$(\mathbb{Z})$
$K = \mathbb{F}_q(T)$	$(\mathbb{Q})$
$K_\infty = \mathbb{F}_q((T^{-1}))$	$(\mathbb{R})$
$C = \hat{\bar{K}}_\infty$	$(\mathbb{C})$
$\Omega = C - K_\infty$	$(\mathbb{H})$
$\mathrm{GL}_2(A)$	$(\mathrm{SL}_2(\mathbb{Z}))$
$\mathfrak{p} = \langle \pi \rangle$	(ℓ)
π prime of degree d	
$\mathbb{F}_\mathfrak{p} = A/\mathfrak{p}$	$(\mathbb{Z}/\ell\mathbb{Z})$
$x = \zeta_x \left(\frac{1}{T}\right)^{v_\infty(x)} u_x \in K$	
$[i] = T^{q^i} - T$	
$g_\mathfrak{p}$, the genus of $X_0(\mathfrak{p})$	

Contents

Chapter 1

Introduction

Because of the deep similarities between the ring $\mathbb{Z}$ and the ring $\mathbb{F}_q[T]$, for q a power of a prime and T an indeterminate, there is a rich and fruitful interaction between results that can be obtained by taking one or the other as a basic object of study. In this work we shall be interested in studying the so-called Drinfeld setting, which offers for function fields certain constructions playing roles analogous to those played by modular forms, modular curves, and elliptic curves in the classical number field setting.

Let A be the ring of functions on a curve X regular outside of a chosen closed point denoted by ∞. We will say that such a ring is an *affine ring*. While many of the constructions we will present here are available for any such A, the analogy is most rich and most developed when X is chosen to be $\mathbb{P}^1$ and the chosen point is defined over the base field $\mathbb{F}_q$. For this reason, we will restrict our attention to this situation in the present work.

In this set up, the field $K = \mathbb{F}_q(T)$ plays the role of the field $\mathbb{Q}$, the field $K_\infty = \mathbb{F}_q((T^{-1}))$ plays the role of $\mathbb{R}$ and the completion of the algebraic closure of K_∞, denoted here by C, plays the role of the field $\mathbb{C}$. For the convenience of the reader we collect the notations specific to this setting on page iv, along with the analogous classical object when appropriate. More generally, whenever we speak of an affine ring A, we will write K for its field of fractions and K_∞ for the completion of K at ∞.

In the first Section of this Introduction, we give a brief overview of the history of the Drinfeld setting and of some of the results that have been obtained. In the second Section, we present the classical results that guided and inspired the work presented here. In the last Section we present an overview of what this work contains and summarize the results that we will obtain.

1.1 The Drinfeld setting

In 1935, Carlitz [36] began the study of an exponential-like function defined over the field $\mathbb{F}_q(T)$. (This function is the Carlitz exponential function defined in equation (2.2) below.) His work led him naturally to consider certain polynomials which he showed shared many properties with the classical cyclotomic polynomials [37]. However, it was not until 1974 that Hayes [31] completed the work of showing how Carlitz's theory could be used to generate the maximal abelian extension of $\mathbb{F}_q(T)$ and to construct the reciprocity law homomorphism of class field theory.

In that same year Drinfeld [14] published a paper in which he established function field analogues of the Kronecker-Weber theorem, the main theorem of complex multiplication, and the Eichler-Shimura relation. His proof introduced objects now called Drinfeld modules, which are an important object of study in this work and are defined in a very explicit manner in Section 2.1. For now, we will say that for an affine ring A, a Drinfeld A-module is a group scheme locally isomorphic with $\mathbb{G}_a$ and provided with an A-action. While it seems that Drinfeld was unaware of Carlitz's work, it is now understood that when $A = \mathbb{F}_q[T]$, the action of A on a certain Drinfeld module of rank 1 (the Carlitz module introduced in equation (2.5)) is exactly given by Carlitz's cyclotomic polynomials.

The significance of Drinfeld modules is that they play for K the role played for $\mathbb{Q}$ by the $\mathbb{Q}$-motives given by semi-abelian schemes. In particular, a major theme of this work is that Drinfeld modules of rank 2 exhibit properties very analogous to those of elliptic curves. This was already obvious in Drinfeld's work: Let F be a quadratic extension of K such that $F \not\subset K_\infty$ (so that F is an "imaginary" quadratic extension), and let H be the Hilbert class field of F. Then if one has a Drinfeld module ϕ of rank 2 defined over H, the torsion points and the j-invariant of ϕ generate the ray class fields of F.

Finally, Drinfeld considers the l-adic cohomology groups of compactifications of moduli spaces of Drinfeld modules of rank 2 with level structure, and exhibits a canonical Galois- and Hecke-equivariant isomorphism between a suitable cohomology group and a certain space of automorphic forms for $\mathrm{GL}_2(\mathbb{A}_K)$. This in turns establishes a correspondence between certain representations of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ and the set of so-called special automorphic representations. For more on this topic and its connection to the local Langlands conjecture, we refer the reader to Deligne and Husemöller's article [12].

In light of the deep connections between the arithmetic properties of K and the Drinfeld modular curves (which are introduced in this work in Chapter 3), it is natural to study their geometry. A useful tool for this purpose are the Drinfeld modular forms, first defined as sections of line bundles on the Drinfeld modular curves by Goss in the late 1970s and the early 1980s ([26], [27], [28]). Goss establishes many of the basic properties of these objects: connections between their algebraic definition and their analytic description as rigid analytic functions, existence of series expansions at the cusps, existence of Hecke operators and Eisenstein series, etc. A perhaps surprising result obtained by Goss is that "multiplicity one" results do not hold here: the Eisenstein series of weight $q-1$ and the Drinfeld discriminant function Δ have the same Hecke eigenvalues for all primes. Goss's work was continued and extended to general affine rings by Gekeler in [22] and in his book [18], where the behavior of Drinfeld modular forms near cusps was studied and consequences about the geometry of the modular curves were derived.

In 1988, Gekeler's *Inventiones* paper [20] introduced two important elements of the theory of Drinfeld modular forms: The first one is a differential operator which he denoted Θ, in analogy to Ramanujan's Θ-operator in the classical case. Gekeler also studied the ring of modular forms "modulo $\mathfrak{p}$", for $\mathfrak{p}$ a prime ideal of $\mathbb{F}_q[T]$, proving some fundamental results analogous to those known in the classical case. The ideas of this paper will feature prominently in this work, and we will present some results on the interplay between differential operators and reduction modulo $\mathfrak{p}$ in Chapter 5.

We note here that by necessity Drinfeld modular forms cannot directly give rise to automorphic forms, as in the classical case, since automorphic forms are functions on adelic groups valued in fields of characteristic zero, while Drinfeld modular forms are functions on a rigid-analytic space valued in a field of finite characteristic. There is a relationship between the two objects, obtained by Gekeler and Reversat in [21]. Roughly speaking, Drinfeld modular forms of a certain type can be thought of as the reduction modulo p (where p is the characteristic of K) of $\mathbb{Z}$-valued automorphic forms. This loss of information explains certain non-classical phenomena such as the failure of the multiplicity one theorem mentioned above and the fact that the action of Hecke operators on Drinfeld modular forms may fail to be semi-simple.

Despite this, it is nevertheless possible to attach a Galois representation to Drinfeld Hecke eigenforms. In 1986, Anderson [3] introduced a higher-dimensional generalization of Drinfeld A-modules called t-motives, and in [29] Goss showed how one can attach function field L-series to these objects. Recently Böckle and Pink [8] generalized the notion of t-motives further to define a theory of A-crystals, which gives a theory similar to the étale setting in the classical case. This has allowed Böckle [7] to interpret Drinfeld modular forms as étale cohomology classes by establishing an Eichler-Shimura isomorphism, and by considering generalized eigenspaces one can thus attach a Galois representation to a cuspidal Drinfeld eigenform. We note here that all representations attached to Drinfeld modular forms are one-dimensional.

1.2 Classical results

In the early 1970s, their work on ℓ-adic Galois representations led Serre and Swinnerton-Dyer to study the algebra of (classical) modular forms modulo ℓ for ℓ a prime of $\mathbb{Z}$. In [49], Swinnerton-Dyer collects some fundamental results: First, he shows that $E_2 \equiv E_{\ell+1} \pmod{\ell}$, for E_2 the false Eisenstein series and $E_{\ell+1}$ the Eisenstein series of weight $\ell+1$, from which it follows that while Ramanujan's Θ-operator $q\frac{d}{dq}$ does not preserve modularity, it preserves modularity modulo ℓ for any prime ℓ. Secondly, he defines a filtration on the algebra of modular forms modulo ℓ and studies how this filtration behaves under the operation of the Θ-operator. Building on these results, Serre [46] defines a notion of ℓ-adic modular forms and shows that modular forms for $\Gamma_0(\ell)$ are ℓ-adic modular forms for $\mathrm{SL}_2(\mathbb{Z})$. His theory allows him to attach ℓ-adic zeta functions to arbitrary totally real extensions of $\mathbb{Q}$. These results have proved to be of fundamental importance in the theory of classical modular forms. For example, they provided the setting for Ahlgren and Boylan's proof [1] that Ramanujan's partition function congruences are the only such congruences of this type. We will present results analogous to those of Serre and Swinnerton-Dyer for the Drinfeld setting in Chapters 5 and 6.

One interesting application of the theory of modular forms modulo ℓ is the study of Weierstrass points on certain modular curves. For a curve X of genus g defined over $\mathbb{C}$, we define a Weierstrass point to be a point P such that there exits a function F on X with a pole of order $\leq g$ at P and regular elsewhere. This defines a finite set of intrinsic geometric points on X. As an example of their significance in geometry, we note that information about Weierstrass points has been recently used to classify certain Hurwitz surfaces by Magaard [39].

Because of the arithmetic significance of the classical modular curves, it is natural to wish to compute their Weierstrass points. This work was started in the 1950s by Petersson [41] and Schoeneberg [45], who studied the Weierstrass points of the modular curves associated to the principal congruence subgroup $\Gamma(N)$, for N an integer. By proving a general result relating the normalizer of the modular group to the Weierstrass points of the associated modular curve, Schoeneberg was able to show that for $N \geq 7$, the cusps of $X(N)$ are Weierstrass points, and that except for three unsettled cases, for $N \geq 9$ the points above i and $\rho = e^{2\pi i/3}$ are also Weierstrass points.

Schoeneberg's method was applied by Lehner and Newman in 1964 [38] to the group $\Gamma_0(N)$ to establish that the cusps of $\Gamma_0(4N)$ and $\Gamma_0(9N)$ are almost always Weierstrass points (their approach does not work in certain cases) and that the fixed points of the Fricke involution W_N are Weierstrass points for all but finitely many $\Gamma_0(N)$. In 1967, a beautiful paper of Atkin's [5] studies ramification at the cusps of covers of the form $X_0(\ell^2 N) \to X_0(\ell N)$ to establish that if N has sufficiently many divisors, the cusps 0 and ∞ are always Weierstrass points, thus leaving open the question of whether one could exhibit an infinite family of curves such that the cusps are not Weierstrass points. Such a family was produced by Ogg in 1978 [40], who used the Deligne-Rapoport model of $X_0(\ell)$ to show that the elliptic curves underlying Weierstrass points of $X_0(\ell)$ have supersingular reduction modulo ℓ. As a consequence of this, the cusps of $X_0(\ell)$ are never Weierstrass points.

Three years later, Rohrlich [42] remarked that a useful tool in the study of Weierstrass points for modular curves might be the Wronskian determinant: For g the genus of the modular curve associated to the congruence group Γ, and $f_1, \ldots f_g$ a basis of weight 2 cusp forms for Γ, consider the form

$$W(f_1, \ldots, f_g) = \begin{vmatrix} f_1(z) & \ldots & f_1^{(g-1)}(z) \\ \vdots & & \vdots \\ f_g(z) & \ldots & f_g^{(g-1)}(z) \end{vmatrix}, \tag{1.1}$$

where $f^{(n)}(z) = \frac{d^n f}{dz^n}$. Then the so-called modular Wronskian W is the multiple of $W(f_1, \ldots, f_g)$ that has leading coefficient 1 in its q-series expansion at ∞. W is modular of weight $g(g+1)$ for Γ and is such that the points in the divisor of $W(z)(dz)^{g(g+1)/2}$ are exactly the Weierstrass points. Although this is not the usual definition, we may further define the Weierstrass weight of a point P, denoted $\mathrm{wt}(P)$, to be the order of vanishing of $W(z)(dz)^{g(g+1)/2}$ at P, since the usual quantity agrees with this.

Following up on this idea, Rohrlich obtains the following theorem in 1985 [43]:

Theorem 1.1. *Let $\ell \geq 23$ and write $\ell + 1 = 12g + r$ for $r = 0$, 6, 8 or 14. Let $W(z)$ be the normalized Wronskian associated to the curve $X_0(\ell)$. Then the Fourier coefficients of $W(z)$ are ℓ-integral and*

$$W(z) \equiv \Delta(z)^{g(g+1)} E_r(z)^g E_{14}^{g(g-1)/2} \pmod{\ell},$$

where E_k is the Eisenstein series of weight k and Δ is the unique normalized cusp form of weight 12 *for* $\mathrm{SL}_2(\mathbb{Z})$.

The two main ingredients of the proof are first the fact that since the moduli problem associated to $X_0(\ell)$ is defined over $\mathbb{Z}$, all the relevant spaces of modular forms have basis with integral Fourier coefficients, and secondly the theorem, proved by Serre in [46], that the forms of weight 2 for $\Gamma_0(\ell)$ correspond modulo ℓ to forms of weight $\ell+1$ for $\mathrm{SL}_2(\mathbb{Z})$.

While interesting, this result did not answer the question of the precise relationship between Weiertrass points on $X_0(\ell)$ and the supersingular locus. It was not until 2003 that Ahlgren and Ono [2] used Rohrlich's work as a starting point to obtain the following beautiful formula:

$$\prod_{Q \in X_0(\ell)} (x - j(Q))^{\mathrm{wt}(Q)} \equiv \prod_{\substack{E/\overline{\mathbb{F}}_\ell \\ E \text{ supersingular}}} (x - j(E))^{g(g-1)} \pmod{\ell}. \tag{1.2}$$

Its significance is the following: The Weierstrass points on $X_0(\ell)$ map to the supersingular elliptic curves in the nicest way possible; each supersingular elliptic curve is the reduction of a Weierstrass point, and the fibers above each supersingular elliptic curve contain the same number of Weierstrass points if each Weierstrass point P is counted with multiplicity $\mathrm{wt}(P)$.

1.3 Our results

We begin this work by reviewing the background we will need. In Chapter 2, we present all of the basic facts about Drinfeld modules and Drinfeld modular forms which will be needed. Then in Chapter 3 we describe the analytic and algebraic constructions of Drinfeld modular curves in some generality, and subsequently restrict our attention to the curves $X_0(\mathfrak{p})$. Finally, in Chapter 4 we present the theory of Drinfeld quasimodular forms and some results on the action of the Hasse derivatives on Drinfeld modular forms. At the end of this Chapter we include some integrality and order of vanishing results for these operators which are easy consequences of theorems obtained by Bosser and Pellarin.

In Chapter 5 we present some results on the action of the Hasse derivative D_n for n small on the filtration of Drinfeld modular forms which were published in [51]. For $\mathfrak{p}$ an ideal generated by a monic prime polynomial π of degree d, E the false Drinfeld Eisenstein series, g_d the Drinfeld Eisenstein series of weight $q^d - 1$ and ∂ a differential operator which preserves modularity (∂ is introduced in Chapter 4), we show:

Theorem 5.2.

$$E \equiv \partial(g_d) \pmod{\mathfrak{p}}.$$

Thanks to this result we have a Drinfeld modular form which can play the role played by $E_{\ell+1}$ in Swinnerton-Dyer's work [49], and we can obtain the following:

Theorem 5.4. *Let f be a Drinfeld modular form of weight k and type l, and $\mathfrak{p}$ be an ideal generated by a monic prime polynomial π of degree d. If f has rational $\mathfrak{p}$-integral u-series coefficients and is not identically zero modulo $\mathfrak{p}$, then the following are true:*

1. *$D_1(f)$ is the reduction of a modular form modulo $\mathfrak{p}$.*

2. *We have $w_{\mathfrak{p}}(D_1(f)) \equiv w_{\mathfrak{p}}(f) + 2 \pmod{q^d - 1}$ (where we take this to be vacuously true if $w_{\mathfrak{p}}(D_1(f)) = -\infty$). Furthermore $w_{\mathfrak{p}}(D_1(f)) \leq w_{\mathfrak{p}}(f) + q^d + 1$ with equality if and only if $w_{\mathfrak{p}}(f) \not\equiv 0 \pmod{p}$.*

In Chapter 6 we obtain an analogue of Serre's theorem that says that forms of weight 2 for $\Gamma_0(\ell)$ correspond modulo ℓ to forms of weight $\ell+1$ for $\mathrm{SL}_2(\mathbb{Z})$. To do so we consider the integrality of the operators $U_{\mathfrak{p}}$ and $V_{\mathfrak{p}}$ on the coefficients of u-series expansions. Our main result is:

Theorem 6.1. *Let $q \geq 3$. There is a one-to-one correspondence between forms of weight 2 and type 1 for $\Gamma_0(\mathfrak{p})$ with rational $\mathfrak{p}$-integral u-series coefficients and forms of weight q^d+1 and type 1 for $\mathrm{GL}_2(A)$ with rational $\mathfrak{p}$-integral u-series coefficients.*

As in Serre's work, this theorem is a corollary of the following theorem:

Theorem 6.2. *Let f be a modular form of weight k and type l for $\Gamma_0(\mathfrak{p})$, with rational u-series coefficients. Then f is a "$\mathfrak{p}$-adic Drinfeld modular form" for $\mathrm{GL}_2(A)$.*

Finally, in Chapter 7 we seek to apply the theory of Drinfeld modular forms to study the Weierstrass points of $X_0(\mathfrak{p})$. We start by reviewing the definition of Weierstrass points in characteristic p, and state a result of Baker's which generalizes Ogg's theorem and can be used to show that the Drinfeld modules underlying the Weierstrass points of $X_0(\mathfrak{p})$ have supersingular reduction modulo $\mathfrak{p}$. We then define the modular Wronskian in our setting, and proceed to show in a specific case an analogue of the theorem of Rohrlich quoted above:

Theorem 7.11. *If p is odd, $\pi \in \mathbb{F}_p[T]$ has degree 3, $\mathfrak{p}$ is the ideal generated by π, and the Wronskian on $X_0(\mathfrak{p})$ is denoted by $W(z)$, then we have*

$$W(z) \equiv (-1)^{(p+1)/2} g^{\frac{p^2(p-1)}{2}} h^{\frac{p^2(p+1)}{2}} \pmod{\mathfrak{p}}.$$

Chapter 2

Drinfeld modules and Drinfeld modular forms

In this Chapter we present basic facts on the theory of Drinfeld modular forms and Drinfeld modules. An excellent standard reference for this material is Gekeler's *Inventiones* paper [20]. For facts on rigid analytic geometry, we refer the reader to [15].

Throughout we will fix q a power of a prime p, and denote by $\mathbb{F}_q$ the finite field with q elements. We will denote by A the ring of polynomials in an indeterminate T, $A = \mathbb{F}_q[T]$, and by K the field $\mathbb{F}_q(T)$, the field of fractions of A. Then for $x \in K$ we may define the order of x at ∞ to be $v_\infty(x) = -\deg(x)$. We will write $K_\infty = \mathbb{F}_q((1/T))$ for the completion of K at its infinite place, and

$$C = \hat{\bar{K}}_\infty$$

for the completed algebraic closure of K_∞. C is thus a complete algebraically closed field of characteristic p. Finally we will also need the analytic space $\Omega = \mathbb{P}^1(C) - \mathbb{P}^1(K_\infty) = C - K_\infty$, which we will call the Drinfeld upper half-plane.

For any $x \in K_\infty^*$, x can be written uniquely as

$$x = \zeta_x \left(\frac{1}{T}\right)^{v_\infty(x)} u_x \tag{2.1}$$

where $\zeta_x \in \mathbb{F}_q^*$, and u_x is such that $v_\infty(u_x - 1) > 0$, or in other words u_x is a 1-unit at

∞. We call ζ_x the *leading coefficient* of x.

2.1 Drinfeld modules

Let Λ be an A-lattice of C, by which we mean a finitely-generated A-submodule having finite intersection with each ball of finite radius contained in C. Consider the following lattice function:

$$e_\Lambda(z) \stackrel{\text{def}}{=} z \prod_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} \left(1 - \frac{z}{\lambda}\right). \tag{2.2}$$

The product converges uniformly on bounded sets in C, thus defining a (rigid) analytic surjective function on C.

If we fix an A-lattice Λ of rank r in C, then for every $a \in A$ there is a unique map ϕ_a^Λ such that for all $z \in C$,

$$\phi_a^\Lambda(e_\Lambda(z)) = e_\Lambda(az).$$

The map

$$\phi^\Lambda : a \mapsto \phi_a^\Lambda$$

defines a ring homomorphism of A into the ring $\mathrm{End}_C(\mathbb{G}_a)$ of additive polynomials over C. In other words, $\mathrm{End}_C(\mathbb{G}_a)$ is the non-commutative ring of polynomials of the form

$$\sum a_i X^{p^i},$$

where multiplication is defined by composition. If we write $\tau = X^q$ and let $C\{\tau\} \subset \mathrm{End}_C(\mathbb{G}_a)$ be the subalgebra of $\mathrm{End}_C(\mathbb{G}_a)$ generated by τ, then in fact Drinfeld shows [14] that ϕ^Λ takes values in $C\{\tau\}$ and for $a \in A$ of degree d we have

$$\phi_a^\Lambda = \sum_{0 \leq i \leq rd} l_i \tau^i \tag{2.3}$$

with $l_0 = a$ and $l_{rd} \neq 0$.

In general, a ring homomorphism $\phi : A \to C\{\tau\}$ that is given by (2.3) is called a *Drinfeld module of rank* r over C. The association $\Lambda \mapsto \phi^\Lambda$ is a bijection of the set of A-lattices of rank r in C with the set of Drinfeld modules of rank r over C. On the side of lattices, we say that Λ and Λ' are *homothetic* if there exists $\lambda \in C^*$ such that $\Lambda' = \lambda \cdot \Lambda$. On the side of Drinfeld modules, we say that two modules ϕ and ψ are *isogenous* if there exists an element $u \in \mathrm{End}_C(\mathbb{G}_a)$ such that $u \circ \phi_a = \psi_a \circ u$ for all $a \in A$. If $u \in C^*$, then we say that the two modules are *isomorphic*. Then we also have that homothety classes of rank r lattices correspond to isomorphism classes of Drinfeld modules of rank r.

More generally, let L be a field over A, for example $L = K$, or $L = A/\mathfrak{p}$, for $\mathfrak{p}$ a prime ideal of A. As before, we may write $L\{\tau\}$ for the subalgebra of $\mathrm{End}_L(\mathbb{G}_a)$ generated by τ. Then a ring homomorphism $\phi : A \to L\{\tau\}$, satisfying

$$\phi_a \stackrel{\text{def}}{=} \phi(a) = \sum_{0 \leq i \leq rd} a_i \tau^i,$$

with $l_0 = a$ and $l_{rd} \neq 0$ for $a \in A$ of degree d, is called a *Drinfeld module of rank* r *over* L. Even more generally, Drinfeld modules may be defined over any A-scheme S, and we refer the reader to Drinfeld's work [14] for the details.

2.1.1 Drinfeld modules of rank 2

We shall be especially interested in Drinfeld modules of rank 2 over a field L, which behave analogously to elliptic curves in the classical case. Such an object is defined by

$$\phi_T = T\tau^0 + g\tau + \Delta\tau^2 \tag{2.4}$$

with $g \in L$, $\Delta \in L$, $\Delta \neq 0$. Writing $a \cdot x = \phi_a(x)$, ϕ gives $\mathbb{G}_a(L)$ a new structure as an A-module. We say that an element x in some field extension of L such that $\phi_a(x) = 0$ is an a-torsion point of ϕ. In general the a-torsion points of a Drinfeld module of rank 2 form a finite A-submodule scheme of $\mathbb{G}_a$ of degree $q^{2\deg a}$.

Consider for a moment the case where the module is defined over C, i.e. g and $\Delta \in C$. Then this module corresponds to a rank 2 lattice $\Lambda_{\omega_1,\omega_2} = A\omega_1 \oplus A\omega_2$. If $\lambda \in C^*$, then $e_{\lambda\cdot\Lambda}(\lambda z) = \lambda e_\Lambda(z)$, so that replacing Λ by a homothetic lattice $\lambda \cdot \Lambda$ will send (g, Δ) to $(\lambda^{1-q}g, \lambda^{1-q^2}\Delta)$. For this reason we define the quantity $j = \frac{g^{q+1}}{\Delta}$, which remains invariant under isomorphism, and call it the j-invariant of the Drinfeld module. Furthermore, replacing Λ by a homothetic lattice, we may restrict our attention to lattices of the form $\Lambda_z = A \oplus Az$. The condition that Λ is an A-lattice translates to having $z \in \Omega$, the Drinfeld upper half-plane. Thus we may view g, Δ and j as functions on Ω, and they will turn out to be Drinfeld modular forms.

Two elements z and z' of Ω define homothetic lattices, and thus isomorphic Drinfeld modules, if and only if there is $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(A)$ such that $z' = \frac{az+b}{cz+d}$. Thus we have that j induces a bijection

$$\mathrm{GL}_2(A)\backslash\Omega \to C.$$

Thus C is a (coarse) moduli space for Drinfeld modules of rank 2 over C. This point of view will be discussed further when Drinfeld modular curves are introduced.

We note here that we may and will define the j-invariant as $\frac{g^{q+1}}{\Delta}$ for a Drinfeld module over an arbitrary field L over A. In general, the j-invariant of a Drinfeld module determines the isomorphism class of the module only over an algebraically closed field.

Let us now consider the case of a Drinfeld module of rank 2 ϕ defined over a finite extension of $\mathbb{F}_\mathfrak{p} = A/\mathfrak{p}$, for $\mathfrak{p}$ a prime ideal generated by a monic polynomial $\pi(T)$ of degree d. We have that

$$\phi_\pi = l_d\tau^d + \ldots + l_{2d}\tau^{2d},$$

i.e. $l_0, \ldots, l_{d-1}$ vanish [16]. As remarked above, the π-torsion points of ϕ form a finite A-submodule scheme of $\mathbb{G}_a$ of degree q^{2d}. In the particular case at hand, because of the vanishing of the coefficients $l_0, \ldots, l_{d-1}$, this group scheme is never reduced, and the group of π-torsion points defined over $\overline{\mathbb{F}}_\mathfrak{p}$ is either 0 or isomorphic to $\mathbb{F}_\mathfrak{p}$. In the case where there are no non-trivial π-torsion points, we say that ϕ, or sometimes its j-invariant, is *supersingular.*

It is known that for a given prime ideal $\mathfrak{p}$, there are only finitely many supersingular j-invariants "in characteristic $\mathfrak{p}$" [16], and that if d, the degree of π, is odd, then $j = 0$ is always supersingular.

Finally, consider a Drinfeld module ϕ defined over K, defined by $\phi_T = T\tau^0 + g\tau + \Delta\tau^2$. When g and Δ are $\mathfrak{p}$-integral, we can talk about the *reduction of* ϕ *modulo* $\mathfrak{p}$, which is the module defined by $\tilde{\phi}_T = \tilde{T}\tau^0 + \tilde{g}\tau + \tilde{\Delta}\tau^2$ over $\mathbb{F}_\mathfrak{p}$, where $\tilde{}$ denotes the reduction modulo $\mathfrak{p}$. If $\tilde{\phi}$ is supersingular, we say that ϕ is *supersingular at* $\mathfrak{p}$.

2.1.2 The parameter at ∞

An important Drinfeld module is Carlitz's module ρ of rank 1, first studied by Carlitz in [36], and defined by:

$$\rho_T = T\tau^0 + \tau. \tag{2.5}$$

Under the correspondence between Drinfeld modules and lattices mentioned above, this Drinfeld module corresponds to a certain rank 1 A-lattice $L = \tilde{\pi}A$, where the *Carlitz period* $\tilde{\pi} \in K_\infty(\sqrt[q-1]{-T})$ is defined up to a $(q-1)$th root of unity. We choose one such

$\tilde{\pi}$ and fix it for the remainder of this work.

Consider now the function

$$u(z) \stackrel{\text{def}}{=} \frac{1}{e_L(\tilde{\pi}z)} \tag{2.6}$$

for $L = \tilde{\pi}A$ and $\tilde{\pi}$ the Carlitz period. Then we have that for any $c > 1$, u induces an isomorphism of the set

$$A\backslash\{z \in \Omega \mid \inf_{x \in K_\infty} |z - x| \geq c\}$$

with a pointed ball $B_r\backslash\{0\}$ for r small. Thus $u(z)$ can be used as a "parameter at infinity", analogously to $q = e^{2\pi iz}$ in the classical case. This in fact is not quite correct from a rigid analytic perspective, and the subtleties involved here will be discussed in much more detail when we discuss the analytic structure of the Drinfeld modular curves in Chapter 3.

2.2 Drinfeld modular forms

For an ideal $\mathfrak{a}$ of A, define the *principal congruence subgroup* $\Gamma(\mathfrak{a})$ to be $\ker(\mathrm{GL}_2(A) \to \mathrm{GL}_2(A/\mathfrak{a}))$. Then a *congruence subgroup* of $\mathrm{GL}_2(A)$ is a subgroup Γ such that $\Gamma(\mathfrak{a}) \subset \Gamma \subset \mathrm{GL}_2(A)$ for some ideal $\mathfrak{a}$ of A. We also note that any such Γ acts on the Drinfeld upper half plane via $\gamma \cdot z = \frac{az+b}{cz+d}$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

For a congruence subgroup Γ, we define its *cusps* to be the finite set $\Gamma\backslash\mathbb{P}^1(K)$. We will see in Chapter 3 that these equivalence classes are in one-to-one correspondence with the points needed to compactify the open Drinfeld modular curve $\Gamma\backslash\Omega$. By a *local parameter* or *uniformizer* at a cusp, we will mean (a root of) a uniformizer for the compactified modular curve at such a point (see Chapter 3), and denote such a uniformizer by t.

Definition 2.1. *Let Γ be a congruence subgroup of* $\mathrm{GL}_2(A)$*. A function $f : \Omega \to C$ is called a* Drinfeld modular form of weight k and type l for Γ, *where $k \geq 0$ is an integer and l is a class in $\mathbb{Z}/(\#\det\Gamma)$, if*

1. *for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, $f(\gamma z) = (\det\gamma)^{-l}(cz+d)^k f(z)$;*

2. *f is rigid analytic on Ω (see Section 3.1.1);*

3. *f is analytic at the cusps of Γ: at each cusp f has an expansion $f(z) = F(t(z))$ where F is a power series with a positive radius of convergence and t is a local parameter at this cusp.*

Remark 2.2. *There are no non-trivial modular forms of weight k and type l if $k \not\equiv 2l$* (mod $\#(\Gamma \cap Z(K))$)*, where $Z(K)$ is the center of* $\mathrm{GL}_2(K)$*.*

Remark 2.3. *If at a given cusp f has an expansion $f(z) = F(t(z))$ and $F = \sum_{i \geq i_0} a_i t^i$ with $a_{i_0} \neq 0$, we say that f* vanishes to order i_0 *at this cusp. If f is a modular form for Γ, we say that f is a cusp form if it vanishes to order at least 1 at each of the cusps of Γ, and that f is a double cusp form if it vanishes to order at least 2 at each of the cusps of Γ. We will see in Chapter 3 that these notions are well-defined if not quite correct from a rigid analytic perspective.*

We will denote the (finite dimensional) vector space of modular forms of weight k and type l for a congruence group Γ by $M_{k,l}(\Gamma)$, the subspace of cusp forms by $M^1_{k,l}(\Gamma)$, and the subspace of double cusp forms by $M^2_{k,l}(\Gamma)$.

For $\gamma \in \mathrm{GL}_2(K)$ we have that $\det\gamma \in K^*$. By (2.1), we can write

$$\det\gamma = \zeta_{\det\gamma}\left(\frac{1}{T}\right)^{v_\infty(\det\gamma)} u_{\det\gamma}.$$

For simplicity we write

$$\zeta_{\det \gamma} = \zeta_\gamma.$$

We define a *slash operator* for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(K)$ on a modular form of weight k and type l by

$$f|_{k,l}[\gamma] = \zeta_\gamma^l \left(\frac{\det \gamma}{\zeta_\gamma}\right)^{k/2} (cz+d)^{-k} f(\gamma z). \tag{2.7}$$

Note that for $\gamma \in \mathrm{GL}_2(A)$ we have that $\det \gamma = \zeta_\gamma$; thus if f is modular of weight k and type l for Γ and $\gamma \in \Gamma$, then $f|_{k,l}[\gamma] = f$. We also record for future reference that since for x, y and $z \in K^*$ such that $z = xy$, we have $\zeta_z = \zeta_x \zeta_y$, it follows that for α, β and $\gamma \in \mathrm{GL}_2(K)$ such that $\gamma = \alpha\beta$, we have:

$$f|_{k,l}[\gamma] = f|_{k,l}[\alpha] f|_{k,l}[\beta].$$

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2.3 Drinfeld modular forms for $\mathrm{GL}_2(A)$

In this section we will consider the case $\Gamma = \mathrm{GL}_2(A)$, and present some fundamental results. In this case the modular curve only has one cusp, ∞, and the function $u(z)$ constructed in Section 2.1.2 can and will be used as a parameter at ∞.

For k a positive integer and $z \in \Omega$, Goss defines in [28] an Eisenstein series of weight $q^k - 1$ by:

$$g_k \stackrel{\text{def}}{=} (-1)^{k+1} \tilde{\pi}^{1-q^k} L_k \sum_{\substack{a,b \in A \\ (a,b) \neq (0,0)}} \frac{1}{(az+b)^{q^k-1}}, \tag{2.8}$$

where L_k is the least common multiple of all monics of degree k, so that

$$L_k = (T^q - T) \dots (T^{q^k} - T),$$

and $\tilde{\pi}$ is the Carlitz period fixed above. These series converge and thus define rigid analytic functions on Ω. They should be considered the analogues of the classical Eisenstein series, and they can be shown to be modular of weight $q^k - 1$ and type 0 for $\mathrm{GL}_2(A)$. Furthermore, it is shown in [20] that with this normalization each g_k has integral u-series coefficients.

Another modular form for $\mathrm{GL}_2(A)$ which will be important in this paper is the Poincaré series of weight $q+1$ and type 1, first defined by Gerritzen and van der Put in [24, page 304]. Let H be the subgroup

$$\left\{ \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \right\} \subset \mathrm{GL}_2(A)$$

and as usual

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(A).$$

Then we may define a series

$$h \stackrel{\text{def}}{=} \sum_{\gamma \in H \backslash \mathrm{GL}_2(A)} \frac{\det \gamma \cdot u(\gamma z)}{(cz+d)^{q+1}}. \tag{2.9}$$

Using the properties of the function $u(z)$, this series can be shown to in fact define a Drinfeld modular form of weight $q+1$ and type 1. It is shown in [20] that h also has integral u-series coefficients. As remarked above, the coefficient Δ of a Drinfeld module of rank 2 as in (2.4) can be considered a function on Ω, and is a Drinfeld modular form. In fact, we have $\tilde{\pi}^{1-q^2}\Delta = -h^{q-1}$, but this will not be used here.

We also note that g_1 is a multiple of the modular form g which appears in equation (2.4). More precisely, $g_1 = \tilde{\pi}^{1-q} g$. From now on, to simplify notation we will denote g_1 simply by g, or alternatively we will replace the old g by its normalization. Since we shall not need to think of g as the coefficient of a Drinfeld module again, this normalization should create no confusion.

It is a well-known fact (see for example [20]) that the graded C-algebra of Drinfeld modular forms of all weights and all types for $\mathrm{GL}_2(A)$ is the polynomial ring $C[g, h]$ (where each Drinfeld modular form corresponds to a unique isobaric polynomial). The first few terms of the u-series expansion of g and h are:

$$g = 1 - (T^q - T)u^{q-1} - (T^q - T)u^{(q-1)(q^2-q+1)} + (T^q - T)u^{(q-1)q^2} + \dots \tag{2.10}$$

and

$$h = -u - u^{1+(q-1)^2} + (T^q - T)u^{1+(q-1)q} - u^{1+(q-1)(2q-2)} + \dots . \tag{2.11}$$

We thus note that g is not a cusp form, and h has a single zero at ∞. Furthermore, since the u-series expansions of g and h both have integral coefficients, every space of modular forms for $\mathrm{GL}_2(A)$ has a basis of forms with integral coefficients.

In [13], the authors remark that the fact that the algebra of Drinfeld modular forms is generated by g and h implies the following: For k a positive integer and l a class in $\mathbb{Z}/(\#\det \mathrm{GL}_2(A)) = \mathbb{Z}/(q-1)$, define the unique functions $\mu(k, l)$ and $\gamma(k, l)$ such that $\mu(k, l) \equiv l \pmod{q-1}$, $0 \leq \gamma(k, l) \leq q$, and $k = \mu(k, l)(q + 1) + \gamma(k, l)(q - 1)$. Then to every Drinfeld modular form of weight k and type l for $\mathrm{GL}_2(A)$ one can associate a unique polynomial $F(f, x) \in C[x]$ such that

$$f = g^{\gamma(k,l)} h^{\mu(k,l)} F(f, j) \tag{2.12}$$

where j is the (normalized) j-invariant, $j = \frac{g^{q+1}}{-h^{q-1}}$.

Finally, we will need the following computation:

Proposition 2.4. *For $q \geq 3$, the dimension of the space of modular forms of weight q^d+1 and type 1 for $\mathrm{GL}_2(A)$ is equal to $g_{\mathfrak{p}}+1$, and the dimension of its subspace of double cusp forms is $g_{\mathfrak{p}}$, where*

$$g_{\mathfrak{p}} \stackrel{def}{=} \begin{cases} \frac{q(q^{d-1}-1)}{q^2-1} & \text{if } d \text{ is odd,} \\ \frac{q^2(q^{d-2}-1)}{q^2-1} & \text{if } d \text{ is even.} \end{cases} \tag{2.13}$$

Remark 2.5. *The use of $g_{\mathfrak{p}}$ for the quantity above will be explained later: it will turn out to be the genus of the modular curve $X_0(\mathfrak{p})$.*

Proof. We simply use the fact that for a Drinfeld modular form of weight k and type l for $\mathrm{GL}_2(A)$, there are constants $c_n \in C$ such that

$$f = \sum_{n=0}^{\left\lfloor \frac{\mu(k,l)}{q-1} \right\rfloor} c_n g^{\gamma(k,l)+n(q+1)} h^{\mu(k,l)-n(q-1)}, \tag{2.14}$$

where $\gamma(k,l)$ and $\mu(k,l)$ are the integers defined immediately before this Proposition. In the case $k = q^d+1$ and $l = 1$, we have

$$\mu(q^d+1,1) = 1 + \frac{q(q^{d-1}-1)}{q+1} \qquad \text{and} \qquad \gamma(q^d+1,1) = 0$$

if d is odd, and

$$\mu(q^d+1,1) = 1 + \frac{q^2(q^{d-2}-1)}{q+1} \qquad \text{and} \qquad \gamma(q^d+1,1) = q$$

if d is even.

In any case the dimension of the space is given by

$$\left\lfloor \frac{\mu(q^d+1,1)}{q-1} \right\rfloor + 1,$$

which is equal to

$$\frac{q(q^{d-1}-1)}{q^2-1} + 1 \qquad \text{if } d \text{ is odd,}$$

and

$$\frac{q(q^{d-2}-1)}{q^2-1}+1 \qquad \text{if } d \text{ is even.}$$

To obtain only double cusp forms, we restrict sum (2.14) to $n < \left\lfloor \frac{\mu(k,l)}{q-1} \right\rfloor$. $\square$

2.4 Modular forms modulo $\mathfrak{p}$

In this Section we fix a monic prime polynomial $\pi(T) \in A$ of degree d and denote by $\mathfrak{p}$ the principal ideal that it generates. For $x \in K$, we write $v_{\mathfrak{p}}(x)$ for the valuation of x at $\mathfrak{p}$. As a general rule, we will continue to denote the reduction homomorphism $A \to \mathbb{F}_{\mathfrak{p}} \stackrel{\text{def}}{=} A/\mathfrak{p}$ and everything derived from it by a tilde $a \mapsto \tilde{a}$.

Definition 2.6. *Let $f = \sum_{i=0}^{\infty} c_i u^i$ be a formal series with $c_i \in K$. Then we define the* valuation of f at $\mathfrak{p}$ *to be*

$$v_{\mathfrak{p}}(f) = \inf_i v_{\mathfrak{p}}(c_i).$$

For two formal series $f = \sum a_i u^i$ and $g = \sum b_i u^i$, we write $f \equiv g \pmod{\mathfrak{p}^m}$ if $v_{\mathfrak{p}}(f - g) \geq m$.

An important fact first proved in [20] which we will need later is that if $\mathfrak{p}$ is an ideal generated by a prime polynomial of degree d, we have $g_d \equiv 1 \pmod{\mathfrak{p}}$. In addition, this is the only relation upon reducing modulo $\mathfrak{p}$. To make this more precise, recall that $\mathbb{F}_{\mathfrak{p}} = A/\mathfrak{p}$, and write $M_{\mathfrak{p}}$ for the ring of modular forms of all weights and types for $\mathrm{GL}_2(A)$ having u-series coefficients in K with denominators prime to $\mathfrak{p}$ and

$$\tilde{M} \stackrel{\text{def}}{=} \{\tilde{f} \in \mathbb{F}_{\mathfrak{p}}[[u]] \mid \exists f \in M_{\mathfrak{p}} \text{ such that } f \equiv \tilde{f} \pmod{\mathfrak{p}}\}$$

for the $\mathbb{F}_{\mathfrak{p}}$-algebra of Drinfeld modular forms modulo $\mathfrak{p}$. Finally, write $A_d[X, Y]$ for the unique isobaric polynomial such that $g_d = A_d[g,h]$; we have that $A_d \in A[X,Y]$ [20]. Then we have:

Theorem 2.7 ([20])**.** *Let* $\tilde{}$ *denote reduction modulo* $\mathfrak{p}$*. Assuming the notation and hypotheses above,*

$$\tilde{M} \cong \mathbb{F}_{\mathfrak{p}}[X,Y]/(\tilde{A}_d(X,Y)-1).$$

Remark 2.8. *It follows from this that if f is of weight k and f' is of weight k' with $f \equiv f' \pmod{\mathfrak{p}}$, then $k \equiv k' \pmod{q^d-1}$.*

In fact we have more: For f any u-series with rational $\mathfrak{p}$-integral coefficients, define its *filtration modulo* $\mathfrak{p}$, denoted $w_{\mathfrak{p}}(f)$, to be the smallest integer k such that there exists a modular form f' of weight k for $\mathrm{GL}_2(A)$ such that $f \equiv f' \pmod{\mathfrak{p}}$. We write $w_{\mathfrak{p}}(f) = -\infty$ if $f \equiv 0 \pmod{\mathfrak{p}}$. As in the classical case, there is a deep connection between supersingular Drinfeld modules in characteristic $\mathfrak{p}$ and forms with lower filtration than weight.

To make this more precise, define the *Drinfeld supersingular locus* to be the following polynomial:

$$S_{\mathfrak{p}}(x) = \prod_{\substack{\phi \text{ defined over } \overline{\mathbb{F}}_{\mathfrak{p}} \\ \phi \text{ supersingular}}} (x - j(\phi)).$$

Recall from equation (2.12) the polynomial $F(g_d, x)$ attached to the Eisenstein series of weight q^d-1 and the integers $\gamma(q^d-1,0)$ defined in the paragraph immediately preceding equation (2.12). Then Gekeler shows [20]

$$S_{\mathfrak{p}}(x) \equiv x^{\gamma(q^d-1,0)} F(g_d, x) \pmod{\mathfrak{p}},$$

where $\gamma(q^d-1,0)$ is 0 if d is even and 1 if d is odd. This result can be refined:

Proposition 2.9 ([13])**.** *Assuming the notation above, let f be a Drinfeld modular form for $\mathrm{GL}_2(A)$ of weight k and type l with rational $\mathfrak{p}$-integral u-series coefficients and finite filtration $w_{\mathfrak{p}}(f)$. Define $\alpha = \frac{k-w_{\mathfrak{p}}(f)}{q^d-1}$ and $a = \left\lfloor \frac{\alpha\gamma(q^d-1,0)q+\gamma(w_{\mathfrak{p}}(f),l)}{q+1} \right\rfloor$. Then the polynomial $x^a F(f,x)$ is divisible by $S_{\mathfrak{p}}(x)^{\alpha}$ in $\mathbb{F}_{\mathfrak{p}}[x]$.*

Chapter 3

Drinfeld modular curves

We now turn our attention to Drinfeld modular curves, and more specifically to the family $X_0(\mathfrak{p})$. The facts contained in the first Section of this Chapter are true with appropriate modifications more generally for affine rings A, [18]. For facts on rigid analytic spaces, again the first chapter of [15] provides much of what will be needed here.

3.1 General facts

Set-theoretically, the set of C-points of a Drinfeld modular curve is simply the set of equivalence classes of points of $\Gamma\backslash(\Omega \cup \mathbb{P}^1(K))$ for Γ a congruence subgroup of $\mathrm{GL}_2(A)$. In this Section we quickly go over the results presented in [21] describing the algebraic and analytic structures on this set.

3.1.1 Analytic structure on Ω

It is clear that the Drinfeld upper half-plane Ω has the structure of a rigid analytic space, since it is obtained by removing the compact set $\mathbb{P}^1(K_\infty)$ from the space $\mathbb{P}^1(C)$. In fact, a pure covering of Ω can be given explicitly:

Recall that throughout we have had a distinguished valuation $v_\infty(x) = -\deg(x)$ on

K with local parameter T^{-1}. The completion of K at this place is the field K_∞ which we have encountered already, and we will denote the ring of integers of K_∞ by O_∞. The valuation gives rise to an absolute value normalized so that $|x| = q^{\deg x}$. To give a rigid analytic structure to Ω we will construct a pure covering for it. Consider the sets $\{D_n\}$ (not to be confused with the Hasse derivative given in Chapter 4) where for n an integer we write D_n for the set of $z \in C$ such that

$$q^{-n-1} \leq |z| \leq q^{-n}$$

and

$$|z - cT^{-n}| \geq q^{-n}, |z - cT^{-n-1}| \geq q^{-n-1} \text{ for all } c \in \mathbb{F}_q^*.$$

It can be shown that $D_n \subset \Omega$. Further, each D_n is an affinoid space over K_∞.

Write now

$$D_{(n,x)} = x + D_n, \qquad \text{for } x \in K_\infty$$

and define the set of indices

$$I = \{(n, x) \mid \text{ for } n \in \mathbb{Z}, x \text{ runs through a set of representatives of } K_\infty/T^{-n-1}O_\infty\}.$$

Then $\Omega = \cup_{(n,x)\in I} D_{(n,x)}$, and $\{D_{(n,x)}\}$ is a pure covering of Ω.

Remark 3.1. *The rigid analytic functions on $D_{(n,x)}$ are the functions of the form:*

$$\begin{aligned} f(z) = \sum_{i=0}^{\infty} a_i(T^n(z-x))^i + \sum_{i=1}^{\infty} b_i(T^{n+1}(z-x))^{-i} \\ + \sum_{c\in\mathbb{F}_q^*}\sum_{i=1}^{\infty} d_{i,c}(T^n(z-x)-c)^{-i} + \sum_{c\in\mathbb{F}_q^*}\sum_{i=1}^{\infty} e_{i,c}(T^{n+1}(z-x)-c)^{-i} \end{aligned}$$

with a_i, b_i, $d_{i,c}$, $e_{i,c}$ in C and

$$\lim_{i\to\infty} a_i = \lim_{i\to\infty} b_i = \lim_{i\to\infty} d_{i,c} = \lim_{i\to\infty} e_{i,c} = 0.$$

Let

$$R\colon \Omega \to \tilde{\Omega}$$

be the analytic reduction associated to this covering. Then $\tilde{\Omega}$ is a scheme over $\mathbb{F}_q$ which is locally of finite type. Each irreducible component L of $\tilde{\Omega}$ is isomorphic with $\mathbb{P}^1(\mathbb{F}_q)$, and meets exactly $q+1$ other components L'. The intersections are ordinary double points which are rational over $\mathbb{F}_q$, and conversely each $\mathbb{F}_q$-rational point s of L determines a component L' that meets L exactly at s. We say that two components that meet non-trivially are adjacent. The intersection graph of $\tilde{\Omega}$ is the graph T whose vertices are given by the components L of $\tilde{\Omega}$, and whose (oriented) edges are (L, L'), for L, L' adjacent components; this graph is a $(q+1)$-regular tree. The action of the group $\mathrm{GL}_2(K_\infty)$ on Ω by fractional linear transformations gives rises to an action on the sets $D_{(n,x)}$, and thus on this tree.

Changing our focus for a moment, one may define the Bruhat-Tits tree $\mathcal{T}$ of the group $\mathrm{PGL}_2(K_\infty)$ in the following manner: Define an O_∞-lattice in K_∞^2 to be a rank two O_∞-submodule of K_∞^2. As usual, two lattices L and L' are homothetic if there is $x \in K_\infty^*$ such that $L' = xL$. Homothety is an equivalence relation and we write $[L]$ for the class of lattice equivalent to L. Two classes $[L]$ and $[L']$ are adjacent if there exists $L'' \in [L']$ such that $L'' \subset L$ and L/L'' has length 1 as an O_∞-module. Then the Bruhat-Tits tree $\mathcal{T}$ of $\mathrm{PGL}_2(K_\infty)$ is the graph whose vertices are the classes $[L]$ and whose edges are $([L],[L'])$ for two adjacent classes $[L]$ and $[L']$, and $\mathcal{T}$ is a $(q+1)$-regular tree. Finally, $\mathrm{GL}_2(K_\infty)$ acts on the left on the set of equivalence classes of lattices by $\gamma \cdot L = L\gamma^{-1}$, and this action gives rise to a left action on the Bruhat-Tits tree.

In fact, there exists a canonical $\mathrm{GL}_2(K_\infty)$-equivariant identification between the two graphs described above. We refer the reader to [21] for a precise description of this map, which relies on the construction of a building map $\lambda\colon \Omega \to \mathcal{T}(\mathbb{R})$ whose description would take us too far afield. The importance of this identification is that it will allow us to give a precise analytic description of the cusps of the modular curves which we will discuss. To this end, we need a few more definitions: define a *half-line* in $\mathcal{T}$ to be a subgraph isomorphic to the regular tree of degree 2 with a single terminal vertex. Then an *end* of $\mathcal{T}$ is an equivalence class of half-lines where two half-lines are equivalent if they differ at most by a finite graph. There is a bijection between the set of ends of $\mathcal{T}$ and $\mathbb{P}^1(K_\infty)$, which again is made precise in [21]. It is this correspondence which allows for the construction of a rigid analytic space $\Omega^* = \Omega \cup \mathbb{P}^1(K)$ which will give the compactification of the curves discussed here.

3.1.2 Algebraic structure on Drinfeld modular curves

Denoting as always by Γ a congruence subgroup of $\mathrm{GL}_2(A)$, the action of Γ on the Drinfeld upper half-plane Ω by fractional linear transformations has finite stabilizer for each $z \in \Omega$. It follows thus from basic facts that the quotient $\Gamma\backslash\Omega$ is in fact a rigid analytic space. Moreover, it is smooth of dimension one. In fact, the curve $\Gamma\backslash\Omega$ can be shown to arise from an algebraic curve:

Theorem 3.2 (Drinfeld [14]). *There exists a smooth irreducible affine algebraic curve Y_Γ defined over C such that $\Gamma\backslash\Omega$ and the underlying analytic space Y_Γ^{an} of Y_Γ are canonically isomorphic as analytic spaces over C.*

We note further that the curve Y_Γ is unique up to isomorphism, and in fact defined over a finite abelian extension of K.

This result can be obtained by considering the moduli scheme associated to a certain moduli problem which we now describe: As remarked in Chapter 2, for $a \in A$ and $\mathfrak{a}$ the ideal generated by a, the $\mathfrak{a}$-torsion points of a Drinfeld module of rank 2 ϕ form a finite A-submodule scheme of $\mathbb{G}_a$ of degree $q^{2\deg a}$. If ϕ is defined over an algebraically closed field L containing K, then the $\mathfrak{a}$-torsion defined over L, which we will denote by $\phi[\mathfrak{a}](L)$, is in fact free of rank 2 over $A/\mathfrak{p}$. Therefore an $\mathfrak{a}$-level structure may be defined for ϕ by specifying an A-module isomorphism between $(\mathfrak{a}^{-1}A/A)^2$ and $\phi[\mathfrak{a}](L)$.

These notions may be extended to work for an arbitrary A-scheme S, see [14], or [33] where the authors develop Drinfeld's idea in the classical case. As a result, there is a moduli functor

$$\mathfrak{M}^2(\mathfrak{a}) : S \mapsto \left\{ \begin{matrix} \text{isomorphism classes of Drinfeld modules of} \\ \text{rank 2 over } S \text{ with } \mathfrak{a}\text{-level structure} \end{matrix} \right\}$$

on the category of A-schemes. When $\mathfrak{a}$ is divisible by at least two distinct primes, $\mathfrak{M}^2(\mathfrak{a})$ can be represented by an affine flat A-scheme $M(\mathfrak{a})$ which is smooth of finite type over $\mathbb{F}_q$ and of dimension 1 over $\operatorname{Spec} A$. Furthermore, the fibers of $M(\mathfrak{a})$ over $\operatorname{Spec} A$ are smooth away from $\mathfrak{a}$. By taking quotients by finite groups, one may construct a scheme $M(\mathfrak{a})$ for arbitrary $\mathfrak{a}$ (including $\mathfrak{a} = 1$) which is a coarse moduli scheme for $\mathfrak{M}^2(\mathfrak{a})$. Again by taking finite quotients, one may also construct coarse moduli schemes for more general moduli problems than that of classifying Drinfeld modules with full $\mathfrak{a}$-level structure. We will discuss one such problem in the next Section.

Then the content of Theorem 3.2 is that for every congruence subgroup Γ the curve $\Gamma\backslash\Omega$ can be identified with the set of C points of the coarse moduli scheme of some moduli problem, thus producing the algebraic curve we seek. For example, as described in Section 2.1.1, the set of isomorphism classes of Drinfeld modules of rank 2 over C is given by $\mathrm{GL}_2(A)\backslash\Omega$, which gives an isomorphism $\mathrm{GL}_2(A)\backslash\Omega \cong M(1)(C)$. Thus

$\mathrm{GL}_2(A)\backslash\Omega$ is algebraically the affine curve $\mathbb{A}^1$.

3.1.3 Analytic structure on Drinfeld modular curves

Since the curves Y_Γ are affine and smooth over C, from algebraic geometry we know that for each Y_Γ there exists a smooth projective model which we will denote X_Γ. As mentioned above, we will use the Bruhat-Tits tree $\mathcal{T}$ to give an analytic description of X_Γ.

Considering the action of Γ on $\mathcal{T}$, Serre shows in [47] that $\Gamma\backslash\mathcal{T}$ is the edge-disjoint union of a finite graph and a finite number of half-lines. In fact, an end of $\mathcal{T}$ gives rise to an end of $\Gamma\backslash\mathcal{T}$ if and only if it K-rational (recall that the ends of $\mathcal{T}$ are in bijection with $\mathbb{P}^1(K_\infty)$). Thus the ends in $\Gamma\backslash\mathcal{T}$ correspond bijectively to the elements of the finite set $\Gamma\backslash\mathbb{P}^1(K)$.

Using the building map λ, one can make precise the notion that compactifying $\Gamma\backslash\Omega$ can be done by filling in the missing points

$$X_\Gamma(C) - Y_\Gamma(C) \cong \{\text{ends of } \Gamma\backslash\mathcal{T}\} \cong \Gamma\backslash\mathbb{P}^1(K).$$

As before, we call the elements of this finite set *the cusps of* Γ. Filling in these points is done analytically by specifying a uniformizer at each cusp.

3.1.4 u-series expansions of Drinfeld modular forms

As promised in Chapter 2, we are now prepared to discuss the role of the function $u(z)$ as a parameter at ∞. We tackle here the case where $\Gamma = \mathrm{GL}_2(A)$, leaving the case of $\Gamma_0(\mathfrak{p})$ for later and the general discussion to [21] or [18].

The set $\mathrm{GL}_2(A)\backslash\mathbb{P}^1(K)$ consists of a single element, and we choose ∞ as the representative of this element. The stabilizer Γ_∞ of ∞ in $\mathrm{GL}_2(A)$ is the set of all upper-triangular matrices. This set contains a maximal subgroup Γ_∞^u:

$$\Gamma_\infty^u = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in A \right\},$$

and also cyclic transformations $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$ for $a, d \in \mathbb{F}_q^*$. The size of the group of these cyclic transformations is $q-1$, the size of $\mathbb{F}_q^*$.

Now writing

$$\Omega_c = \{z \in \Omega \mid \inf_{x \in K_\infty} |z - x| \geq c\},$$

we recall that $u(z)$ identifies $A\backslash\Omega_c$ with a pointed ball $B_r\backslash\{0\}$ of radius r for some small r. It can be shown that there is a constant c_0 such that for $c \geq c_0$ and $\gamma \in \mathrm{GL}_2(A)$, $\Omega_c \cap \gamma(\Omega_c) \neq \emptyset$ implies that $\gamma \in \Gamma_\infty$. Thus for such a c,

$$\begin{array}{ccccc} B_{r^{q-1}}\backslash\{0\} & \cong & \Gamma_\infty\backslash\Omega_c & \hookrightarrow & \mathrm{GL}_2(A)\backslash\Omega \\ u^{q-1}(z) & \leftarrow & z & \rightarrow & z \end{array}$$

is an open immersion of analytic spaces. Thus $u(z)^{q-1}$ is a uniformizer at ∞ for $\mathrm{GL}_2(A)\backslash\Omega$.

The subtlety involved in defining the u-series expansion of a Drinfeld modular form is that we allow them to have a non-trivial type l, and thus they are not invariant under the full Γ_∞, but rather only under Γ_∞^u. This is why in general a Drinfeld modular form of non-trivial type will have a u-series expansion rather than a u^{q-1}-series expansion. A similar phenomenon may happen for any congruence subgroup for which Γ_∞^u is strictly contained in Γ_∞. Thus our definition of the order of vanishing of a Drinfeld modular form at ∞ is not strictly correct, even though it is convenient. This will not cause any problems in this work until Chapter 7, where we shall have to be more careful.

There is also a second subtlety that comes into play. For a general congruence subgroup Γ, to discuss the behavior of a function f at a cusp $s \in \Gamma\backslash\mathbb{P}^1(K)$, one first fixes an element $\gamma \in \mathrm{GL}_2(K)$ such that $\gamma \cdot \infty = s$. Then the holomorphy properties and order of vanishing of f at s are the properties of $f \circ \gamma$ at ∞, and do not depend on the choice of s in its equivalence class modulo Γ and on the choice of γ sending ∞ to s. However, for t a parameter at ∞ for the group Γ, one might wish to define the t-series expansion of f at s as that of $f \circ \gamma$ at ∞. This is not well-defined, as the coefficients of the expansion will depend on the choice of s and γ.

To remove any ambiguity, in the case of $\mathrm{GL}_2(A)$ we once and for all declare that the expansion of f at ∞ is its u-series expansion, with u as described above.

3.1.5 Drinfeld modular forms as geometric objects

As in the classical case, we have that for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$,

$$\frac{d(\gamma z)}{dz} = \det \gamma (cz + d)^{-2},$$

so that for f a modular form for Γ of weight 2 and type 1, the differential form $f(z)dz$ is Γ-invariant. A short computation, presented in [21], shows that it descends to a holomorphic differential form on X_Γ if f is a double cusp form, or in other words if f has at least a double zero at each cusp. The need for a double zero is explained by the fact that if t is a parameter at a cusp, we have

$$dz = -t^{-2}dt.$$

Since GAGA theorems hold for rigid analytic curves, we have the following theorem:

Theorem 3.3. *The map $f \mapsto f(z)dz$ identifies the space of double cusp forms of weight 2 and type 1 for Γ to the space of regular differential forms on X_Γ.*

From this theorem it follows that the dimension of the space of double cusp forms of weight 2 and type 1 for Γ is g_Γ, where g_Γ is the genus of the curve X_Γ. Furthermore, it follows by a standard argument that all spaces of Drinfeld modular forms of a fixed weight and type for a congruence group Γ are finite-dimensional.

3.2 The Drinfeld modular curves $X_0(\mathfrak{p})$

We fix again a monic prime polynomial $\pi(T) \in A$ of degree d and denote by $\mathfrak{p}$ the principal ideal that it generates. In this Section we restrict our attention to the family of Drinfeld modular curves attached to the congruence subgroups

$$\Gamma = \Gamma_0(\mathfrak{p}) \stackrel{\text{def}}{=} \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(A) \mid c \equiv 0 \pmod{\mathfrak{p}} \right\}.$$

In this case, $\# \det \Gamma_0(\mathfrak{p}) = q - 1$. From now on, we will denote the affine curve $Y_{\Gamma_0(\mathfrak{p})}$ by $Y_0(\mathfrak{p})$ and the projective curve $X_{\Gamma_0(\mathfrak{p})}$ by $X_0(\mathfrak{p})$ to coincide with classical notation.

3.2.1 The moduli problem

As remarked above, every congruence subgroup corresponds to a certain moduli problem for Drinfeld modules of rank 2. The problem attached to $\Gamma_0(\mathfrak{p})$ classifies Drinfeld modules of rank 2 with a distinguished A-submodule scheme of degree q^d contained in the $\mathfrak{p}$-torsion. Writing $M(\mathfrak{p})$ again for the moduli scheme associated to the full $\mathfrak{p}$-level structure, we can obtain a (coarse) moduli scheme for our problem by considering

$M_0(\mathfrak{p}) = B\backslash M(\mathfrak{p})$, for

$$B = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \subset \mathrm{GL}_2(A/\mathfrak{p}).$$

We have the following theorem from [19], where j is the j-invariant and $j_{\mathfrak{p}}(z) = j(\pi z)$:

Theorem 3.4.

- *$M_0(\mathfrak{p}) \to \operatorname{Spec} A$ is smooth away from $\mathfrak{p}$.*
- *$M_0(\mathfrak{p})$ is the normalization of $\operatorname{Spec} A[j]$ in its function field $K(j, j_{\mathfrak{p}})$.*
- *$M_0(\mathfrak{p}) \times_A K = Y_0(\mathfrak{p})$.*
- *If d is even, $M_0(\mathfrak{p})$ is regular. If d is odd, $M_0(\mathfrak{p})$ has a singularity on the fiber above $\mathfrak{p}$ at the supersingular j-invariant $j = 0$, and is otherwise regular. The singularity is of type A_q.*

Remark 3.5. *In fact, one may consider this moduli problem for "generalized Drinfeld modules" to obtain the same results for $\bar{M}_0(\mathfrak{p})$, a scheme over $\operatorname{Spec} A$ whose generic fiber is $X_0(\mathfrak{p})$.*

The last part of the Theorem requires a careful study of the moduli problem "in characteristic $\mathfrak{p}$". To obtain it, Gekeler [19] shows that the special fiber of $X_0(\mathfrak{p})$ is given by two copies of $X_0(1)$ intersecting transversally at the supersingular points and interchanged by the Fricke involution $W_{\mathfrak{p}}$. At the level of Drinfeld modules, if ϕ is a Drinfeld module and H is a A-submodule scheme of degree q^d contained in the $\mathfrak{p}$-torsion of ϕ, so that (ϕ, H) is a point of $M_0(\mathfrak{p})$, then $W_{\mathfrak{p}}(\phi, H) = (\phi/H, \phi[\mathfrak{p}]/H)$, where $\phi[\mathfrak{p}]$ denotes the $\mathfrak{p}$-torsion of ϕ.

3.2.2 Main properties of $X_0(\mathfrak{p})$

We gather in this Section the various facts we will need about the curve $X_0(\mathfrak{p})$.

From Theorem 3.4 above, we have that $X_0(\mathfrak{p})$ is defined over K with function field $K(j, j_{\mathfrak{p}})$. In fact, because the moduli problem associated to $\Gamma_0(\mathfrak{p})$ is defined over A, the space of holomorphic differentials on $X_0(\mathfrak{p})$ has a basis that is defined over A. Therefore, the space of Drinfeld double cusp forms of weight 2 and type 1 for $\Gamma_0(\mathfrak{p})$ has a basis of forms with integral coefficients. It also follows from such considerations that Drinfeld modular forms on $\Gamma_0(\mathfrak{p})$ with rational u-series coefficients have bounded denominators.

From its action on pairs (ϕ, H), we can also see that the Fricke involution $W_{\mathfrak{p}}$ is K-rational. We note here that the analytic avatar of $W_{\mathfrak{p}}$ is the action of the matrix $\left(\begin{smallmatrix} 0 & -1 \\ \pi & 0 \end{smallmatrix}\right)$ on Ω.

We have that the genus of $X_0(\mathfrak{p})$, which we denote by $g_{\mathfrak{p}}$, is given by

$$g_{\mathfrak{p}} = \begin{cases} \frac{q(q^{d-1}-1)}{q^2-1} & \text{if } d \text{ is odd,} \\[2ex] \frac{q^2(q^{d-2}-1)}{q^2-1} & \text{if } d \text{ is even.} \end{cases}$$

(As promised, this is the same $g_{\mathfrak{p}}$ that appears in Proposition 2.4.) This fact can be obtained either by relating $g_{\mathfrak{p}}$ to $h_1(\Gamma_0(\mathfrak{p}))\backslash\mathcal{T})$ as in [23], or by working directly on the Drinfeld modular curve as in [22].

From [22], we also note that $X_0(\mathfrak{p})$ has two cusps, given for example by 0 and ∞, and that both of these cusps are K-rational. From the same source, we have that $X_0(\mathfrak{p})(C)$ has $\frac{q^d+1}{q+1}$ elliptic points, all of which have ramification index $q+1$ over $X(1)$. For an elliptic point P such that $\tau \in \Omega$ is a representative of P in the Drinfeld upper-half plane we write e for the order of the stabilizer of τ in $\widetilde{\Gamma_0}(\mathfrak{p})$. Then e is a divisor of $q+1$ [21].

Finally, we have from [19] that when $d = 3$, $X_0(\mathfrak{p})$ is hyperelliptic with involution

$W_{\mathfrak{p}}$.

3.2.3 Expansions at the cusps

For our results we will need to speak of the behavior of Drinfeld modular forms for $\Gamma_0(\mathfrak{p})$ at its cusps. Considering first the cusp ∞, its stabilizer Γ_∞ in $\Gamma_0(\mathfrak{p})$ is again the set of all-upper triangular matrices in $\mathrm{GL}_2(A)$. Because of this, the same argument as in Section 3.1.4 shows that u^{q-1} is a parameter at ∞, and that modular forms for $\Gamma_0(\mathfrak{p})$ have a u-series expansion at ∞. As in the case of $\mathrm{GL}_2(A)$, we fix once and for all that the expansion of f at ∞ is its u-series expansion.

To fix a well-defined choice of u-series expansion at the other cusp, we fix 0 as the representative of the other equivalence class, and the matrix

$$W_{\mathfrak{p}} = \begin{pmatrix} 0 & -1 \\ \pi & 0 \end{pmatrix}$$

as the matrix sending ∞ to 0. Thus the u-series expansion of a Drinfeld modular form of weight k and type l at 0 is defined to be that of the form

$$f|_{k,l}[W_{\mathfrak{p}}] = \pi^{k/2}(\pi z)^{-k} f\left(\frac{-1}{\pi z}\right)$$

at ∞.

Chapter 4

Hyperderivatives and Quasimodular forms

In this Chapter we present the theory necessary to study the action of differential operators on the algebra of Drinfeld modular forms. These operators will not preserve modularity, which naturally leads us to consider a larger set of functions on Ω, the Drinfeld quasimodular forms. Throughout we will use "analytic" to mean "rigid analytic". We will say that a function f on Ω is "analytic at ∞" to mean that there are constants a_i such that $f(z) = \sum_{i=0}^{\infty} a_i u(z)^i$ for $|z|_i$ large. In this case we write $\mathrm{ord}_\infty(f)$ for the least $i \geq 0$ such that $a_i \neq 0$.

4.1 Drinfeld quasimodular forms

Definition 4.1. *An analytic function $f : \Omega \to C$ is called a* Drinfeld quasimodular form of weight k, type l, and depth m for $\mathrm{GL}_2(A)$, *where $k \geq 0$ and $m \geq 0$ are integers and l is a class in $\mathbb{Z}/(q-1)$, if there exist analytic functions $f_1, f_2, \ldots, f_m$ which are A-periodic and analytic at infinity such that for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(A)$, we have*

$$f(\gamma z) = (\det \gamma)^{-l}(cz+d)^k \sum_{j=0}^{m} f_j(z) \left(\frac{c}{cz+d}\right)^j .$$

For a given quasimodular form $f \neq 0$, the weight, type and polynomial $\sum_{j=0}^{m} f_j(z)X^j$ are uniquely determined by f as shown in [10]. Furthermore, as can be seen by choosing γ to be the identity matrix, we necessarily have $f = f_0$. Finally, every modular form is a quasimodular form of depth 0, and vice-versa.

An important example of a Drinfeld quasimodular form is the function E introduced in [20]:

$$E \stackrel{\text{def}}{=} \frac{1}{\tilde{\pi}} \sum_{\substack{a \in \mathbb{F}_q[T] \\ a \text{ monic}}} \left(\sum_{b \in \mathbb{F}_q[T]} \frac{a}{az+b} \right),$$

which can be shown to be of weight 2, type 1 and depth 1. Its importance is reflected in the fact that the graded C-algebra of Drinfeld quasimodular forms of all weights, types and depths is the polynomial ring $C[g, h, E]$, where each form corresponds to a unique isobaric polynomial.

For a more in depth discussion of Drinfeld quasimodular forms, we refer the interested reader to the seminal papers [10] and [11] by Bosser and Pellarin.

4.2 Higher derivatives

In [50], Uchino and Satoh consider the action of the Hasse derivatives on analytic functions on Ω. We present here the results we need from their paper without proof.

We will use the fact that C is a complete field with a non-Archimedean dense valuation (which we recall is the unique extension of $v_\infty(x) = -\deg(x)$ from K to C) and that Ω is an open set. We will work in this section with analytic functions on Ω and denote the space of these functions by $\mathrm{An}(\Omega)$. For $f \in \mathrm{An}(\Omega)$ such that $f = \sum_{i=0}^{\infty} c_{i,w}(z-w)^i$

in a neighborhood of w, we define the n^{th} hyperderivative of f at $w \in \Omega$ to be

$$\mathfrak{D}_n(f)(w) = c_{n,w}. \tag{4.1}$$

As remarked above, this is simply the Hasse derivative.

For our purposes, it will be important that our differential operator preserves K-rationality of the u-series coefficients, which $\mathfrak{D}_n$ does not. However, the operator

$$D_n \stackrel{\text{def}}{=} \frac{1}{(-\tilde{\pi})^n}\mathfrak{D}_n \tag{4.2}$$

does [10], and so we will use this normalized operator.

Remark 4.2. *The operator $-D_1$ was also studied by Gekeler in [20], where it was denoted by Θ, in analogy with Ramanujan's Θ-operator in the classical setting. This is also the notation we adopted in [51], but for consistency throughout this work we will use D_1. This explains the discrepancy in sign between this work and the cited paper in our statement of Proposition 4.5 below.*

We have the following facts, all shown in [50]:

Proposition 4.3. *For $f \in \mathrm{An}(\Omega)$ and $w \in \Omega$ as above, we have:*

1. *Formally, in a neighborhood around w,*

$$D_n f(z) = \frac{1}{(-\tilde{\pi})^n} \sum_{i=0}^{\infty} \binom{i}{n} c_{i,w}(z-w)^{i-n} \tag{4.3}$$

   and this has the same radius of convergence as $\sum_{i=0}^{\infty} c_{i,w}(z-w)^i$ at w. In addition, one may also show that $D_n f$ is analytic.

2. *The system of derivatives $\{D_n\}$ is a* higher derivation*; in other words it satisfies:*

(a) $D_0 f = f$,

(b) D_n *is* C*-linear,*

(c) *for* f *and* g *in* $\mathrm{An}(\Omega)$, $D_n(fg) = \sum_{i=0}^{n} D_i f D_{n-i} g$.

3. *This higher derivation is* iterative*: for all integers* $i \geq 0$ *and* $j \geq 0$*, we have:*

$$D_i \circ D_j = D_j \circ D_i = \binom{i+j}{i} D_{i+j}. \tag{4.4}$$

4. *This higher derivation has a* chain rule property*: For each* $n \geq 1$ *and each* $1 \leq i \leq n$*, there exist maps* $F_{n,i}$ *from* $\mathrm{An}(\Omega)^{n+1-i}$ *to* $\mathrm{An}(\Omega)$ *such that:*

(a) *for* f *and* g *in* $\mathrm{An}(\Omega)$ *such that the composition* $f \circ g$ *is defined, we have*

$$D_n(f \circ g) = \sum_{i=1}^{n} F_{n,i}(D_1 g, \ldots, D_{n+1-i} g)(D_i f) \circ g, \tag{4.5}$$

(b) *and if* $n \geq 2$*, then* $F_{n,1}$ *is a* C*-linear map.*

We note here that since the D_n's are iterative and using Lucas' theorem, we have that

$$D_n = \frac{1}{n_0! \ldots n_s!} D_{p^s}^{n_s} \circ \ldots \circ D_p^{n_1} \circ D_1^{n_0}, \tag{4.6}$$

for $n = n_s p^s + \cdots + n_1 p + n_0$ the representation of n in base p, with $0 \leq n_j \leq p-1$ for each j, and where the exponent of n_j on D_{p^j} denotes the n_j-fold composition.

As remarked at the beginning of this Chapter, the D_n's do not preserve modularity, but they do preserve quasimodularity, as shown in [10]. For our purposes we shall only need this weaker version of their more general theorem:

Proposition 4.4. *Let* f *be a modular form of weight* k *and type* l *for* $\mathrm{GL}_2(A)$*. Then for all* $n \geq 0$ *and* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(A)$*, we have*

$$D_n f(\gamma z) = (cz+d)^k (\det \gamma)^{-l} \sum_{j=0}^{n} \binom{n+k-1}{j} \frac{D_{n-j} f}{(-\tilde{\pi})^j} \left(\frac{c}{cz+d}\right)^j. \tag{4.7}$$

In other words, the function $D_n f$ is a quasi-modular form of weight k, type l and depth n.

4.3 Computational tools

The action of D_n quickly becomes difficult to compute explicitly as n grows. A better-behaved operator was defined by Serre in the classical case (see [32]), and we will use its analogue in the Drinfeld setting. Let n and d be non-negative integers. The n^{th} *Serre's operator of degree* d is defined by the formula:

$$\partial_n^{(d)} f = D_n f + \sum_{i=1}^{n} (-1)^i \binom{d+n-1}{i} (D_{n-i} f)(D_{i-1} E). \tag{4.8}$$

In [11], the authors show that $\partial_n^{(k)}$ sends modular forms of weight k and type l to modular forms of weight $k+2n$ and type $l+n$.

Because of the difficulty of the computations, we will be especially interested in D_n and $\partial_n^{(d)}$ for $1 \leq n < p$. We first observe that

$$D_1 = -\frac{1}{\tilde{\pi}} \frac{d}{dz} = u^2 \frac{d}{du},$$

where u is the parameter at ∞ defined in Chapter 2, and for $1 \leq n < p$, $D_1^n = n! D_n$, where the exponent on D_1 denotes n-fold composition.

For simplicity we will denote the operator $\partial_1^{(k)}$ by ∂, considering it as a differential operator of weight 2 on the graded algebra of Drinfeld modular forms. Then

$$\partial(f) = D_1(f) - kEf,$$

where k is the weight of f.

We have the following:

Proposition 4.5 ([20]).

1. *Let f_i be Drinfeld modular forms of weight k_i, then $\partial(f_1 f_2) = \partial(f_1) f_2 + f_1 \partial(f_2)$.*

2. *$\partial(g) = -h$ and $\partial(h) = 0$.*

This proposition allows us to compute the action of ∂ on all Drinfeld modular forms, since g and h generate the algebra of Drinfeld modular forms. Furthermore, since $D_n(E) = E^{n+1}$ for $1 \leq n < p$, a tedious but easy computation shows that for a Drinfeld modular form f of weight k, we have

$$\partial^n f = n! \partial_n^{(k)} f \tag{4.9}$$

for $1 \leq n < p$, where again the exponent on ∂ on the lefthand side denotes n-fold composition of the ∂ operator. This relation in fact holds for $p \leq n < q$ as well, which simply implies that the n-fold composition of ∂ beyond ∂^{p-1} is identically zero, as expected in characteristic p.

4.4 Integrality results

For $i \in \mathbb{N}$, write $[i] = T^{q^i} - T$, the product of all monic prime polynomials of degree dividing i, $d_i = [1]^{q^{i-1}} \cdots [i-1]^q [i]$, the product of all monics of degree i, and $d_0 = 1$. In [10], Bosser and Pellarin obtain the following result on the action of the D_n's on the u-series coefficients of quasimodular forms:

Proposition 4.6. *Let $f \in \mathrm{An}(\Omega)$ be analytic at ∞ with u-series expansion $f(z) =$*

$\sum_{i\geq 0} a_i u^i$. *Then for all* $n \geq 0$ *we have* $D_n(f(z)) = \sum_{i\geq 2} b_{n,i} u^i$, *where*

$$b_{n,i} = \sum_{r=1}^{i-1} (-1)^{n+r} \binom{i-1}{r} \left(\sum_{\substack{n_1,\ldots,n_r \geq 0 \\ q^{n_1}+\cdots+q^{n_r}=n}} \frac{1}{d_{n_1}\cdots d_{n_r}} \right) a_{i-r}.$$

From this explicit formula we can clearly see that

Corollary 4.7. *For* $n < q^e$, *the operator* D_n *preserves* $\mathfrak{p}$*-integrality of the* u*-series coefficients for all* $\mathfrak{p}$ *generated by a prime polynomial of degree* $\geq e$.

Proof. If $n < q^e$, then we have $n_j < e$ for each n_j appearing in the sum defining the $b_{n,i}$'s above. Since d_{n_j} is only divisible by primes of degree $\leq n_j$, for $n < q^e$ D_n introduces only denominators of degree $< e$. $\square$

From this it easily follows that

Corollary 4.8. *Suppose that* $f \equiv f' \pmod{\mathfrak{p}}$ *for* $\mathfrak{p}$ *generated by a prime of degree* d. *Then* $D_n(f) \equiv D_n(f') \pmod{\mathfrak{p}}$ *for* $n < q^d$.

4.5 Vanishing results

Proposition 4.9. *Let* $w \in \Omega$ *and* f *be analytic on* Ω. *Then* $\operatorname{ord}_w D_n(f) \geq \operatorname{ord}_w(f) - n$, *with equality if and only if* $\binom{\operatorname{ord}_w(f)}{n} \not\equiv 0 \pmod{p}$ *when* $n \leq \operatorname{ord}_w(f)$.

Proof. Locally around w, f can be written as $\sum_{i=\operatorname{ord}_w(f)}^{\infty} a_i(z-w)^i$, with $a_{\operatorname{ord}_w(f)} \neq 0$. As remarked before, locally around w we have

$$D_n(f) = \sum_{i=\operatorname{ord}_w(f)}^{\infty} \binom{i}{n} a_i (z-w)^{i-n}.$$

Thus if $n \leq \text{ord}_w(f)$, we have that $\text{ord}_w D_n(f) \geq \text{ord}_w(f) - n$ with equality if and only if $\binom{\text{ord}_w(f)}{n} \not\equiv 0 \pmod{p}$, as claimed. If $n > \text{ord}_w(f)$, then all we can say is that $\text{ord}_w D_n(f) \geq 0$, which in any case ensures that $\text{ord}_w D_n(f) \geq \text{ord}_w(f) - n$. □

Proposition 4.10. *Let f be analytic at ∞, then $\text{ord}_\infty D_n(f) \geq \max(\text{ord}_\infty(f) + 1, 2)$. If $\text{ord}_\infty(f) \geq 1$, we have $\text{ord}_\infty D_n(f) = \text{ord}_\infty(f) + 1$ if and only if $\text{ord}_\infty(f) \not\equiv 0 \pmod{p}$.*

Proof. This follows easily from Proposition 4.6. □

Chapter 5

Drinfeld modular forms modulo $\mathfrak{p}$

In this Chapter we continue to write $\pi(T)$ for a fixed monic prime polynomial in A of degree d and we denote by $\mathfrak{p}$ the principal ideal that it generates. We will also write $\mathbb{F}_{\mathfrak{p}} = A/\mathfrak{p}$, and $M_{\mathfrak{p}}$ for the graded algebra of modular forms of all weights and types for $\mathrm{GL}_2(A)$ having u-series coefficients in K with denominators prime to $\mathfrak{p}$

Following [49], in this Chapter we find it convenient to adopt the following notations: Throughout, we will use $\tilde{}$ to denote the reduction modulo $\mathfrak{p}$. For $A_{\mathfrak{p}}$ the localization of the ring A at $\mathfrak{p}$, if f is a function which has a u-series expansion $\sum_{i=0}^{\infty} a_i u^i$ such that every a_i is in $A_{\mathfrak{p}}$, then $\tilde{f}$ will denote the formal power series $\sum_{i=0}^{\infty} \tilde{a}_i u^i$. Similarly, if $\phi(X,Y)$ is a polynomial in $A_{\mathfrak{p}}[X,Y]$, then $\tilde{\phi}(X,Y)$ will denote the polynomial in $\mathbb{F}_{\mathfrak{p}}[X,Y]$ obtained from ϕ by reducing its coefficients modulo $\mathfrak{p}$. Naturally we will wish to evaluate these polynomials at the formal power series in u corresponding to $\tilde{g}$ and $\tilde{h}$, and denote by $\tilde{\phi}(\tilde{g},\tilde{h})$ the element of $\mathbb{F}_{\mathfrak{p}}[[u]]$ obtained from this polynomial by substitution. As a consequence of this notation, if f is a Drinfeld modular form in $M_{\mathfrak{p}}$, there is a unique polynomial ϕ such that $f = \phi(g,h)$, and $\tilde{f} = \tilde{\phi}(\tilde{g},\tilde{h})$. Finally, motivated by the derivation ∂ defined in Chapter 4, we define a derivation, also denoted ∂, on $A_{\mathfrak{p}}[X,Y]$ and $\mathbb{F}_{\mathfrak{p}}[X,Y]$ by setting $\partial(X) = -Y$ and $\partial(Y) = 0$. The operator D_1 described earlier analogously extends from $A_{\mathfrak{p}}[[u]]$ to $\mathbb{F}_{\mathfrak{p}}[[u]]$.

Since $M_{\mathfrak{p}}$ contains the elements g and h, we have the following composition of homomorphisms:

$$\begin{aligned} A_{\mathfrak{p}}[X,Y] \xrightarrow{\sim} \mathbb{F}_{\mathfrak{p}}[X,Y] &\xrightarrow{\epsilon} \mathbb{F}_{\mathfrak{p}}[[u]] \\ (X,Y) &\mapsto (\tilde{g},\tilde{h}) \end{aligned} \tag{5.1}$$

(where we recall that the tilde denotes the "reduction modulo $\mathfrak{p}$" homomorphism). Consequently we will assign weight $q-1$ to X and weight $q+1$ to Y. Here we repeat a theorem from [20] that was stated in Chapter 2 in order to make more precise a result about the reduction of g_d modulo $\mathfrak{p}$:

Theorem 5.1. *Let $A_d \in A[X,Y]$ be the polynomial defined by $A_d(g,h) = g_d$. Assuming the notation and hypotheses above, the following are true:*

1. *$\tilde{A}_d(X,Y)$ is square-free.*

2. *$\tilde{M} \cong \mathbb{F}_{\mathfrak{p}}[X,Y]/(\tilde{A}_d(X,Y)-1)$.*

We recall that as a consequence of this Theorem, if $f_i \in M_{\mathfrak{p}}$ is of weight k_i for $i=1$, 2 and $f_1 \equiv f_2 \pmod{\mathfrak{p}}$, then $k_1 \equiv k_2 \pmod{q^d-1}$. Thus $\tilde{M}$ has a natural grading by $\mathbb{Z}/(q^d-1)\mathbb{Z}$. As in Chapter 2 we will denote by $w_{\mathfrak{p}}(f)$ the filtration of f, which is defined to be the smallest integer k such that there exists a Drinfeld modular form of weight k congruent to f modulo $\mathfrak{p}$, with as before the convention that the form 0 has weight $-\infty$.

5.1 Derivatives and reduction modulo $\mathfrak{p}$

Theorem 5.2. *Let $B_d \in A[X,Y]$ be the polynomial defined by $B_d(g,h) = \partial(g_d)$. Assuming the notation and hypotheses above, the following are true:*

1. *$\tilde{B}_d(X,Y)$ shares no common factor with $\tilde{A}_d(X,Y)$.*

2. *We have $E \equiv \partial(g_d) \pmod{\mathfrak{p}}$.*

Proof. For the proof of the first fact, let a be an irreducible factor of $\tilde{A}_d$ over $\overline{\mathbb{F}}_{\mathfrak{p}}[X,Y]$ and write $\tilde{A}_d = a \cdot b$. Since $\tilde{A}_d$ is square-free, a does not divide b. We have:

$$\tilde{B}_d = \partial(\tilde{A}_d) = \partial(a)b + a\partial(b)$$

and a divides $\tilde{B}_d$ if and only if a divides $\partial(a)$. Since a must be isobaric, we can have either $a = X$, $a = X^{q+1} + cY^{q-1}$ for some nonzero c in the algebraic closure of $\mathbb{F}_{\mathfrak{p}}$, or $a = Y$. In the first two cases, we have respectively that $\partial(a) = Y$ and $\partial(a) = X^q Y$, so a does not divide $\partial(a)$.

The third possibility (in which case a divides $\partial(a)$) does not happen. In other words, Y does not divide $\tilde{A}_d$ for any d. This can be shown using induction on d and the recursive formula, proven in [20],

$$\tilde{A}_d = \tilde{A}_{d-1}X^{q^{d-1}} + (T^{q^{d-1}} - T)\tilde{A}_{d-2}Y^{q^{d-2}(q-1)} \tag{5.2}$$

with $\tilde{A}_0 = 1$ and $\tilde{A}_1 = X$. By (5.2), if Y does not divide $\tilde{A}_{d-1}$, then Y does not divide $\tilde{A}_d$. Obviously Y does not divide $\tilde{A}_1$, so Y does not divide $\tilde{A}_d$ for any d.

For the proof of the second fact, it suffices to note that since $g_d \equiv 1 \pmod{\mathfrak{p}}$, $D_1(g_d) \equiv 0 \pmod{\mathfrak{p}}$ and $\partial(g_d) = D_1(g_d) - (q^d - 1)Eg_d \equiv E \pmod{\mathfrak{p}}$. $\square$

We will also need a result on modular forms that have lower filtration than weight:

Proposition 5.3. *Let f be a Drinfeld modular form in $M_{\mathfrak{p}}$ of weight k and type l with $\tilde{f} \neq 0$, and write $f = \phi(g,h)$. Then $w_{\mathfrak{p}}(f) < k$ if and only if $\tilde{A}_d | \tilde{\phi}$.*

Proof. Suppose that f' is of weight strictly less than f and $f \equiv f' \pmod{\mathfrak{p}}$. Write $f' = \psi(g,h)$ with $\psi \in A_{\mathfrak{p}}[X,Y]$. Then

$$\tilde{\phi} = c(\tilde{A}_d - 1) + \tilde{\psi}$$

for some polynomial $c \in \mathbb{F}_{\mathfrak{p}}[X,Y]$. Writing $c = \sum_{i=0}^{n} c_i$ as a sum of its isobaric components with c_i of weight strictly less than c_{i+1}, we have that

$$\tilde{\phi} = c_n \tilde{A}_d, \quad c_0 = \tilde{\psi} \quad \text{and} \quad c_i = c_{i-1}\tilde{A}_d \quad \text{for} \quad i = 1, ...n,$$

and $\tilde{A}_d$ divides $\tilde{\phi}$.

Suppose now that $\tilde{\phi} = \tilde{A}_d \tilde{\psi}$ for some polynomial $\tilde{\psi} \in \mathbb{F}_{\mathfrak{p}}[X,Y]$ which must be isobaric of weight $k - q^d + 1$. Lifting $\tilde{\psi}$ to $\psi \in A[X,Y]$, we have that $f' = \psi(g,h)$ is of weight strictly less than k and $f \equiv f' \pmod{\mathfrak{p}}$. □

5.2 Derivatives and filtration

Theorem 5.4. *Let f be a Drinfeld modular form of weight k and type l, and $\mathfrak{p}$ be an ideal generated by a monic prime polynomial π of degree d. If f has rational $\mathfrak{p}$-integral u-series coefficients and is not identically zero modulo $\mathfrak{p}$, then the following are true:*

1. *$D_1(f)$ is the reduction of a modular form modulo $\mathfrak{p}$.*

2. *We have $w_{\mathfrak{p}}(D_1(f)) \equiv w_{\mathfrak{p}}(f) + 2 \pmod{q^d - 1}$ (where we take this to be vacuously true if $w_{\mathfrak{p}}(D_1(f)) = -\infty$). Furthermore $w_{\mathfrak{p}}(D_1(f)) \leq w_{\mathfrak{p}}(f) + q^d + 1$ with equality if and only if $w_{\mathfrak{p}}(f) \not\equiv 0 \pmod{p}$.*

Proof. By Theorem 5.2, $D_1(f) \equiv \partial(f) g_d + k\partial(g_d) f \pmod{\mathfrak{p}}$, which is a form of weight $k + q^d + 1$ and type $l + 1$. Now without loss of generality assume that f is of weight

$w_{\mathfrak{p}}(f)$. Since $D_1(f)$ is congruent to a form of weight $w_{\mathfrak{p}}(f) + q^d + 1$ it follows that

$$w_{\mathfrak{p}}(D_1(f)) \equiv w_{\mathfrak{p}}(f) + 2 \pmod{q^d - 1}.$$

Furthermore, since f is of weight $w_{\mathfrak{p}}(f)$, then $\tilde{A}_d$ does not divide $\tilde{\phi}$. We have that

$$D_1(\tilde{f}) = \partial(\tilde{\phi}(\tilde{g}, \tilde{h}))\tilde{A}_d(\tilde{g}, \tilde{h}) + w_{\mathfrak{p}}(f)\tilde{B}_d(\tilde{g}, \tilde{h})\tilde{\phi}(\tilde{g}, \tilde{h})$$

so that $D_1(\tilde{f})$ is the image in $\mathbb{F}_{\mathfrak{p}}[[u]]$ of the polynomial

$$\partial(\tilde{\phi})(X, Y)\tilde{A}_d(X, Y) + w_{\mathfrak{p}}(f)\tilde{B}_d(X, Y)\tilde{\phi}(X, Y)$$

under the map ϵ given in (5.1). Since $\tilde{A}_d$ and $\tilde{B}_d$ have no common factors, $\tilde{A}_d$ divides $\partial(\tilde{\phi})\tilde{A}_d + w_{\mathfrak{p}}(f)\tilde{B}_d\tilde{\phi}$ if and only if $w_{\mathfrak{p}}(f) \equiv 0 \pmod{p}$. □

We are now interested in characterizing the action of D_n for $1 \leq n < p$ on the filtration of modular forms. As remarked in Chapter 4, in this range we have $D_1^n = n!D_n$, and $n!$ is a unit in $\mathbb{F}_q[T]$. Since multiplying by a unit does not change the filtration, for simplicity we will consider the n-fold composition of D_1 in place of D_n. To simplify notation we will write D for D_1 and D^n for the n-fold composition of the operator D_1.

For every positive integer k and for p the characteristic of $\mathbb{F}_q[T]$, we define the integer $n(k, p)$ to be the unique integer $0 \leq n(k, p) < p$ such that $k + n(k, p) \equiv 0 \pmod{p}$.

Theorem 5.5. *Let f be a Drinfeld modular form of weight k and type l and $\mathfrak{p}$ be an ideal generated by a monic prime polynomial π of degree d. If f has rational $\mathfrak{p}$-integral u-series coefficients and is not identically zero modulo $\mathfrak{p}$, then*

$$w_{\mathfrak{p}}(D^i(f)) = w_{\mathfrak{p}}(f) + i(q^d + 1) \qquad \text{for } 0 \leq i \leq n(w_{\mathfrak{p}}(f), p).$$

Upon another iteration of D, we have:

$$D^{n(w_{\mathfrak{p}}(f),p)+1}(f) \equiv \partial^{n(w_{\mathfrak{p}}(f),p)+1}(f') \pmod{\mathfrak{p}},$$

for f' of weight $w_{\mathfrak{p}}(f)$ such that $f \equiv f' \pmod{\mathfrak{p}}$.

Proof. If $n(w_{\mathfrak{p}}(f), p) = 0$, the theorem is trivial and we have

$$D(f) \equiv D(f') \equiv \partial(f')g_d + w_{\mathfrak{p}}(f)\partial(g_d)f' \equiv \partial(f') \pmod{\mathfrak{p}}$$

thus proving the additional assertion.

Suppose now that $0 < n(w_{\mathfrak{p}}(f), p) < p$. We define a sequence of modular forms in the following manner, for $0 \leq i \leq n(w_{\mathfrak{p}}(f), p) + 1$:

$$\begin{aligned}
f_0 &= f' \\
f_1 &= \partial(f')g_d + w_{\mathfrak{p}}(f)\partial(g_d)f' \\
f_2 &= \partial(f_1)g_d + (w_{\mathfrak{p}}(f) + 1)\partial(g_d)f_1 \\
&\vdots \\
f_i &= \partial(f_{i-1})g_d + (w_{\mathfrak{p}}(f) + i - 1)\partial(g_d)f_{i-1} \\
&\vdots
\end{aligned}$$

We first claim that

$$f_i \equiv D^i(f) \pmod{\mathfrak{p}} \quad \text{for all } 0 \leq i \leq n(w_{\mathfrak{p}}(f), p) + 1.$$

This follows easily since for any Drinfeld modular form of weight k,

$$D(f) \equiv \partial(f)g_d + k\partial(g_d)f \pmod{\mathfrak{p}}.$$

From this fact, since the weight of each f_i is

$$w_{\mathfrak{p}}(f) + i(q^d + 1) \equiv w_{\mathfrak{p}}(f) + i \pmod{p}$$

it follows that $D(f_i) \equiv f_{i+1} \pmod{\mathfrak{p}}$. To complete the proof of our first claim it suffices now to note that $f_1 \equiv f_2 \pmod{\mathfrak{p}}$ implies $D(f_1) \equiv D(f_2) \pmod{\mathfrak{p}}$.

Now since $f_i \equiv D^i(f)$, of course $w_{\mathfrak{p}}(D^i(f)) = w_{\mathfrak{p}}(f_i)$. For $1 \leq i \leq n(w_{\mathfrak{p}}(f), p)$, a simple induction shows that

$$w_{\mathfrak{p}}(f_{i-1}) = w_{\mathfrak{p}}(f) + (i-1)(q^d+1)$$

is not zero modulo p so that $w_{\mathfrak{p}}(f_i) = w_{\mathfrak{p}}(f) + i(q^d+1)$, as required by part 2 of Theorem 5.4.

Secondly we claim that for each $1 \leq i \leq n(w_{\mathfrak{p}}(f), p)$ and for each $1 \leq j \leq i+1$,

$$\partial^j(f_{i-j+1}) = (w_{\mathfrak{p}}(f) + i)\partial^j(f_{i-j})\partial(g_d) + \partial^{j+1}(f_{i-j})g_d. \tag{5.3}$$

The proof is done by induction on j. For any i in the range and $j = 1$, (5.3) follows by applying ∂ to both sides of the equality defining f_i and remembering that $\partial^2(g_d) = 0$. As an induction step, we suppose that (5.3) is true for $i-1$ and $j-1$, and again by simply applying ∂ we obtain (5.3) for i and j.

Now fix $i = n(w_{\mathfrak{p}}(f), p)$. Then (5.3) becomes

$$\partial^j(f_{n(w_{\mathfrak{p}}(f),p)-j+1}) = \partial^{j+1}(f_{n(w_{\mathfrak{p}}(f),p)-j})g_d \tag{5.4}$$

for $1 \leq j \leq n(w_{\mathfrak{p}}(f), p)$. Using equation (5.4) recursively we obtain that

$$f_{n(w_{\mathfrak{p}}(f),p)+1} = \partial^{n(w_{\mathfrak{p}}(f),p)+1}(f')g_d^{n(w_{\mathfrak{p}}(f),p)+1}$$

Since $D^{n(w_{\mathfrak{p}}(f),p)+1}(f) \equiv f_{n(w_{\mathfrak{p}}(f),p)+1} \pmod{\mathfrak{p}}$ and $g_d \equiv 1 \pmod{\mathfrak{p}}$, our second claim follows. $\square$

5.3 Some applications

5.3.1 Forms of lower filtration than weight

We have the following clear corollary to Theorem 5.5:

Corollary 5.6. *Let f be a Drinfeld modular form in $M_{\mathfrak{p}}$ for $\mathfrak{p}$ an ideal of A generated by a monic prime polynomial, and assume that f is not identically zero modulo $\mathfrak{p}$. Then*

$$D^i(f) \not\equiv 0 \pmod{\mathfrak{p}} \quad \text{for } 1 \leq i \leq n(w_{\mathfrak{p}}(f), p).$$

This corollary can be used to detect forms that have lower filtration than weight. For example, consider any Drinfeld modular form over $\mathbb{F}_{25}[T]$ of weight 1376 and an ideal $\mathfrak{p}$ of A generated by a monic prime of degree 2. Suppose further that it can be shown that $D^3(f) \equiv 0 \pmod{\mathfrak{p}}$. Then it must be the case that f has lower filtration modulo $\mathfrak{p}$ than weight, since a form of filtration 1376 would have $D^i(f) \not\equiv 0 \pmod{\mathfrak{p}}$ for $0 \leq i \leq 4$. One can in fact determine that $w_{\mathfrak{p}}(f) = 128$ in the following manner: As a consequence of Theorem 5.1, we know that the filtration of f must be congruent to 1376 modulo 624. Thus since f has lower filtration than weight, it must have filtration 752 or 128. But if it had filtration 752, the corollary above would say that $D^3(f) \not\equiv 0 \pmod{\mathfrak{p}}$.

5.3.2 Vanishing modulo $\mathfrak{p}$ of coefficients

One can turn the above idea on its head by constructing forms that have lower filtration than weight and using the theory to deduce the vanishing modulo $\mathfrak{p}$ of some of their coefficients. Consider as a toy example the Drinfeld modular form

$$f = (T^q - T)gh^{q+2} + g^{q+2}h^3 = \sum_{i=3}^{\infty} a_i u^i$$

over $\mathbb{F}_q[T]$. It has weight $q^2 + 4q + 1$. If one considers a monic prime polynomial of degree greater than or equal to 3, it is clear that this form has filtration equal to its weight. Thus for such primes we have $n(w_{\mathfrak{p}}(f), p) = p - 1$, from which we may deduce that there is $i \equiv 1 \pmod{p}$ such that $a_i \neq 0$, because $D^{p-1}(f) \neq 0$ and D^{p-1} annihilates all coefficients but those a_i's that have $i \equiv 1 \pmod{p}$. However, for an ideal $\mathfrak{p}$ generated by a prime polynomial of degree 2, the form is congruent to gh^3 modulo $\mathfrak{p}$. (By (5.2), $g_2 = (T^q - T)h^{q-1} + g^{q+1}$, and $g_2 \equiv 1 \pmod{\mathfrak{p}}$ if $\mathfrak{p}$ is generated by a prime polynomial of degree 2.) We will show in Proposition 5.7 that $D^{p-1}(gh^3) = 0$, which implies that $D^{p-1}(f) \equiv 0 \pmod{\mathfrak{p}}$. Thus for each $i \equiv 1 \pmod{p}$, $a_i \equiv 0 \pmod{\mathfrak{p}}$ if $\mathfrak{p}$ is generated by a monic prime polynomial of degree 2.

5.3.3 Support of non-zero coefficients of monomials

Since $D = u^2 \frac{d}{du}$, the operator D^n annihilates the u-series coefficients a_i such that $i \equiv p - i + 1, \ldots p - 1, p \pmod{p}$, for p the characteristic of $\mathbb{F}_q$. Thus the vanishing or non-vanishing of $D^n f$ is related to the indices on which the u-series expansion of f is supported. As a final application of the theorems collected here, we will show a result on the vanishing of coefficients of monomials.

Proposition 5.7. *Let α and β be non-negative integers, and consider the monomial $g^\alpha h^\beta$ which has weight $k = \alpha(q-1) + \beta(q+1)$. Write a for the unique integer such that $0 \leq a < p$ and $a \equiv \alpha \pmod{p}$ and similarly write b for the unique integer such that $0 \leq b < p$ and $b \equiv \beta \pmod{p}$. Then either $0 < b - a < p$ or $b = 0$, in which case*

$$D^{n(k,p)+1}(g^\alpha h^\beta) = 0 \quad \textit{and} \quad D^i(g^\alpha h^\beta) \neq 0 \quad \textit{for } 1 \leq i \leq n(k,p),$$

or $-p < b - a \leq 0$ but $b \neq 0$, in which case

$$D^i(g^\alpha h^\beta) \neq 0 \quad \textit{for } 1 \leq i < p.$$

Proof. First suppose that $\mathfrak{p}$ is an ideal of A generated by a monic prime polynomial of degree 1. Then

$$g^\alpha h^\beta \equiv h^\beta \pmod{\mathfrak{p}}$$

and so

$$w_{\mathfrak{p}}(g^{\alpha}h^{\beta}) = \beta(q+1) \equiv b \pmod{p}.$$

If $b \neq 0$, then $n(w_{\mathfrak{p}}(g^{\alpha}h^{\beta}), p) = p - b$ and by the proof of Theorem 5.5,

$$D^{p-b+1}(g^{\alpha}h^{\beta}) \equiv \partial^{p-b+1}(h^{\beta}) = 0 \pmod{\mathfrak{p}}.$$

Finally, if $b = 0$ then $n(w_{\mathfrak{p}}(g^{\alpha}h^{\beta}), p) = 0$ and by Theorem 5.5 we have $D(g^{\alpha}h^{\beta}) \equiv \partial(h^{\beta}) = 0 \pmod{\mathfrak{p}}$.

Now suppose that $\mathfrak{p}$ is an ideal of A generated by a monic prime polynomial of degree d, where $d > 1$. Then

$$w_{\mathfrak{p}}(g^{\alpha}h^{\beta}) = k \equiv b - a \pmod{p}.$$

We consider two cases:

First suppose that $0 < b - a < p$. Then $n(k, p) = p - b + a$, and

$$D^{p-b+a+1}(g^{\alpha}h^{\beta}) \equiv \partial^{p-b+a+1}(g^{\alpha}h^{\beta}) = 0 \pmod{\mathfrak{p}}.$$

But then since $p - b + a + 1 \geq p - b + 1$, $D^{p-b+a+1}(g^{\alpha}h^{\beta}) \equiv 0$ modulo every prime ideal in A, and we conclude that $D^{n(k,p)+1}(g^{\alpha}h^{\beta}) = 0$. We also have that for $1 \leq i \leq p - b + a$, $D^{i}(g^{\alpha}h^{\beta}) \neq 0$ since it is not zero modulo $\mathfrak{p}$ for any ideal generated by a prime of degree greater than 1.

Now suppose that $-p < b - a \leq 0$. Then $n(k, p) = a - b$. As above we have that

$$D^{i}(g^{\alpha}h^{\beta}) \neq 0 \quad \text{for } 1 \leq i \leq a - b \tag{5.5}$$

and

$$D^{a-b+1}(g^{\alpha}h^{\beta}) \equiv \partial^{a-b+1}(g^{\alpha}h^{\beta}) \pmod{\mathfrak{p}}.$$

If $b = 0$, then $\partial^{a+1}(g^\alpha h^\beta) = 0$, and the result follows since $D^{n(k,p)+1}(g^\alpha h^\beta) \equiv 0$ modulo every prime ideal of A. If $b \neq 0$, we have

$$D^{a-b+1}(g^\alpha h^\beta) \equiv \partial^{a-b+1}(g^\alpha h^\beta) \neq 0 \pmod{\mathfrak{p}}. \tag{5.6}$$

If $a = p-1$ and $b = 1$, then $a - b + 1 = p - 1$, and so by combining (5.5) and (5.6) the result follows.

Notice now that since $b \neq 0$ and we have already considered the case $a = p - 1$ and $b = 1$, it only remains to consider the cases where $-p + 3 \leq b - a \leq 0$. In any case we have:

$$w_{\mathfrak{p}}(\partial^{a-b+1}(g^\alpha h^\beta)) = (\alpha - a + b - 1)(q-1) + (\beta + a - b + 1)(q+1) \equiv a - b + 2 \pmod{p}.$$

We now would like to apply Theorem 5.5 to $\partial^{a-b+1}(g^\alpha h^\beta)$. Since $-p + 3 \leq b - a \leq 0$, we have $2 \leq a - b + 2 \leq p - 1$. Then

$$n(w_{\mathfrak{p}}(\partial^{a-b+1}(g^\alpha h^\beta)), p) = p - a + b - 2.$$

Applying Theorem 5.5 to $\partial^{a-b+1}(g^\alpha h^\beta)$, we find that

$$D^{a-b+1+i}(g^\alpha h^\beta) \equiv D^i(\partial^{a-b+1}(g^\alpha h^\beta)) \neq 0 \pmod{\mathfrak{p}}$$

for $1 \leq i \leq p - a + b - 2$, or combining with (5.5),

$$D^i(g^\alpha h^\beta) \neq 0 \quad \text{for } 1 \leq i \leq p - 1$$

which is the result we sought. $\square$

Chapter 6

A correspondence

In this Chapter we establish the following:

Theorem 6.1. *Let $q \geq 3$. There is a one-to-one correspondence between forms of weight 2 and type 1 for $\Gamma_0(\mathfrak{p})$ with rational $\mathfrak{p}$-integral u-series coefficients and forms of weight $q^d + 1$ and type 1 for $\mathrm{GL}_2(A)$ with rational $\mathfrak{p}$-integral u-series coefficients.*

This is an analogue of Serre's result [46] that establishes a bijection modulo ℓ between modular forms of weight 2 on $\Gamma_0(\ell)$ with rational, ℓ-integral Fourier series coefficients at ∞, and modular forms of weight $\ell + 1$ on $\mathrm{SL}_2(\mathbb{Z})$ with rational, ℓ-integral Fourier series coefficients. As with Serre's result, this theorem is a direct corollary of a more general result:

Theorem 6.2. *Let f be a modular form of weight k and type l for $\Gamma_0(\mathfrak{p})$, with rational u-series coefficients. Then f is a "$\mathfrak{p}$-adic Drinfeld modular form" for $\mathrm{GL}_2(A)$.*

6.1 Operators and integrality

We begin by introducing two operators relevant to the theory of $\mathfrak{p}$-adic Drinfeld modular forms. As before $\mathfrak{p}$ is a prime ideal generated by a monic prime polynomial $\pi(T)$ of A of degree d. For any analytic function f on Ω that is analytic at ∞ with expansion

$f = \sum_{i=0}^{\infty} a_i u^i$ we define:

$$f|U_{\mathfrak{p}} = \frac{1}{\pi} \sum_{\substack{\lambda \in A \\ \deg \lambda < d}} f\left(\frac{z+\lambda}{\pi}\right),$$

and

$$f|V_{\mathfrak{p}} = f(\pi z).$$

We will show that if the coefficients a_i are integral, then the u-series coefficients of $f|U_{\mathfrak{p}}$ and $f|V_{\mathfrak{p}}$ are also integral and moreover that

$$v_{\mathfrak{p}}(f|U_{\mathfrak{p}}) \geq v_{\mathfrak{p}}(f)$$

and

$$v_{\mathfrak{p}}(f|V_{\mathfrak{p}}) \geq v_{\mathfrak{p}}(f).$$

We first recall some facts about the Carlitz module and fix some notation: Recall equation (2.5), in which we defined the Carlitz module as the rank 1 Drinfeld module given by

$$\rho_T = T\tau^0 + \tau.$$

Putting $\tau = X^q$, we have

$$\begin{aligned} \rho_\pi &= \pi\tau^0 + \sum_{1 \leq i \leq d-1} \alpha_i \tau^i + \tau^d \\ &= \pi X + \sum_{1 \leq i \leq d-1} \alpha_i X^{q^i} + X^{q^d} \end{aligned}$$

with each α_i in A, and $\alpha_d = 1$ since π is monic. We define the π^{th} inverse cyclotomic polynomial

$$f_\pi(X) = X^{q^d} \rho_\pi(X^{-1}).$$

Obviously f_π is a polynomial with integral coefficients.

Remark 6.3. *We call f_π the π^{th} inverse cyclotomic polynomial because of its connection to the class field theory of the field K. In short, the roots of f_π generate an abelian extension of K with Galois group $(A/\mathfrak{p})^*$, and an explicit subfield of this field is the ray class field of $\mathfrak{p}$. These results are due to Hayes [31], who developed ideas first presented by Carlitz [37].*

We first consider the operator $U_\mathfrak{p}$. This operator was already studied in [9], where the author determines that the $U_\mathfrak{p}$ operator acts in the following manner on the u-series coefficients of analytic functions on Ω (we note that Bosser's result is more general and applies to meromorphic functions with a pole of order less than q^d at infinity, but we will only need the version stated here):

Proposition 6.4. *Let $\mathfrak{p}$ be a prime ideal in A generated by a monic prime polynomial π of degree d, and let f be an analytic function on Ω. Assume that f is analytic at ∞ with a u-series expansion of the form*

$$f = \sum_{i=0}^{\infty} c_i u^i, \qquad c_i \in C.$$

As before we write the Carlitz module at π as $\rho_\pi = \pi\tau^0 + \sum_{1\le i\le d} \alpha_i \tau^i$. Then $f|U_\mathfrak{p}$ has a u-series expansion

$$f|U_\mathfrak{p} = \sum_{j=1}^{\infty} a_j u^j$$

with

$$a_j = \sum_{j\le n\le 1+(j-1)q^d} \sum_{\substack{i\in\mathbb{N}^{d+1} \\ i_0+i_1+\ldots+i_d=j-1 \\ i_0+i_1q+\ldots+i_dq^d=n-1}} \binom{j-1}{i} c_n \alpha_1^{i_1}\ldots\alpha_d^{i_d}\pi^{i_0}.$$

From this explicit result we deduce that $U_\mathfrak{p}$ indeed preserves integrality of the u-series coefficients, since each α_i is integral, and furthermore we have:

Corollary 6.5. *With the same hypotheses as above, we have that*

$$v_{\mathfrak{p}}(f|U_{\mathfrak{p}}) \geq v_{\mathfrak{p}}(f).$$

Proof. This follows from the properties of a non-archimedean valuation, which imply that for each j

$$v_{\mathfrak{p}}(a_j) \geq \min_{j \leq n \leq 1+(j-1)q^d} \{v_{\mathfrak{p}}(c_n)\}.$$

□

We now establish the same properties for the $V_{\mathfrak{p}}$ operator:

Proposition 6.6. *Let $\mathfrak{p}$ be a prime ideal in A generated by a monic prime polynomial π of degree d, and let f be an analytic function on Ω that is analytic at ∞ with u-series expansion of the form*

$$f = \sum_{i=0}^{\infty} c_i u^i, \qquad c_i \in C.$$

Then if each c_i is integral, then so are the u-series coefficients of $f|V_{\mathfrak{p}}$. In addition,

$$v_{\mathfrak{p}}(f|V_{\mathfrak{p}}) \geq v_{\mathfrak{p}}(f).$$

Proof. We have:

$$f|V_{\mathfrak{p}} = \sum_{i=0}^{\infty} c_i u(\pi z)^i,$$

and so we first investigate the u-series expansion of $u(\pi z)$.

By definition, if $L = \tilde{\pi} A$ is the lattice associated to the Carlitz module and $e_L(z)$ is the exponential function associated to it,

$$e_L(\pi z) = \rho_\pi(e_L(z)).$$

Thus we have the straightforward computation:

$$\begin{aligned} u(\pi z) &= \frac{1}{e_L(\pi(\tilde{\pi} z))} \\ &= \frac{1}{\rho_\pi(e_L(\tilde{\pi} z))} \\ &= \frac{1}{\rho_\pi\left(\frac{1}{u(z)}\right)} \\ &= \frac{u(z)^{q^d}}{f_\pi(u(z))}. \end{aligned}$$

Since $f_\pi(0) = 1$, the formal expansion in X for

$$\frac{X^{q^d}}{f_\pi(X)} = X^{q^d} + \text{ higher order terms}$$

has integer coefficients, and $u(\pi z)$ has a formal series expansion in $u(z)$ with integral coefficients.

Thus we have

$$f|V_{\mathfrak{p}} = \sum_{i=0}^{\infty} c_i \left(\frac{u(z)^{q^d}}{f_\pi(u(z))}\right)^i.$$

We note that for j fixed, only a finite number of terms of the right hand side contribute to the coefficient of u^j on the left hand side, and they are all integral. We conclude that $f|V_{\mathfrak{p}}$ has integral u-series expansion.

We also observe that if $v_{\mathfrak{p}}(f) = m$, which implies that $v_{\mathfrak{p}}(c_i) \geq m$ for each i, we have that each of the summands in the coefficient of u^j for fixed j on the left hand side has valuation $\geq m$. We conclude that the coefficient of u^j also has valuation $\geq m$, which in turns implies that $v_{\mathfrak{p}}(f|V_{\mathfrak{p}}) \geq m = v_{\mathfrak{p}}(f)$ and completes the proof. $\square$

We end this section by relating the $V_{\mathfrak{p}}$ operator to the operator $|_{k,l}[W_{\mathfrak{p}}]$ defined by

$$f|_{k,l}[W_{\mathfrak{p}}] = \pi^{k/2}(\pi z)^{-k} f\left(\frac{-1}{\pi z}\right),$$

as before. We have:

Lemma 6.7. *Let f be a modular form for* $\mathrm{GL}_2(A)$ *of weight k and type l. Then*

$$f|_{k,l}[W_{\mathfrak{p}}] = \pi^{k/2} f|V_{\mathfrak{p}}.$$

Proof. We have that

$$W_{\mathfrak{p}} = \begin{pmatrix} 0 & -1 \\ \pi & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}.$$

So that if we let

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2(A), \qquad \text{and} \qquad [\pi] = \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix},$$

we have

$$\begin{aligned} f|_{k,l}[W_{\mathfrak{p}}] &= f|_{k,l}[S]|_{k,l}[\pi] \\ &= f|_{k,l}[\pi] \\ &= \pi^{k/2} f|V_{\mathfrak{p}}, \end{aligned}$$

where the invariance of f under the action of $|_{k,l}[S]$ follows from the fact that f is modular for the full $\mathrm{GL}_2(A)$. □

Remark 6.8. *From this fact, it follows that the action of $|_{k,l}[W_{\mathfrak{p}}]$ preserves integrality of the u-series coefficients if f is modular for* $\mathrm{GL}_2(A)$*. There is no reason for this to be true in general.*

6.2 Additive trace of a Drinfeld modular form

Definition 6.9. *For f a modular form of weight k and type l for Γ a congruence subgroup of* $\mathrm{GL}_2(A)$*, define its* (additive) trace *as*

$$\mathrm{Tr}(f) = \sum_{\gamma \in \Gamma \backslash \mathrm{GL}_2(A)} f|_{k,l}[\gamma].$$

It is clear that $\mathrm{Tr}(f)$ *is independent of the choice of coset representatives for* $\Gamma \backslash \mathrm{GL}_2(A)$*, and that it is a modular form of weight k and type l for* $\mathrm{GL}_2(A)$*.*

We restrict our attention to the case $\Gamma = \Gamma_0(\mathfrak{p})$, where we have:

Lemma 6.10. *Let $\mathfrak{p}$ be an ideal generated by $\pi(T)$, a monic prime ideal. The set*

$$\left\{ \begin{pmatrix} 0 & -1 \\ 1 & \lambda \end{pmatrix} \mid \deg \lambda < \deg \pi \right\},$$

along with the identity, is a complete set of representatives for $\Gamma_0(\mathfrak{p}) \backslash \mathrm{GL}_2(A)$*.*

Proof. Consider

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(A).$$

Suppose that π does not divide c, so that $M \notin \Gamma_0(\mathfrak{p})$. Write $\lambda \equiv c^{-1}d \pmod{\mathfrak{p}}$, picking λ such that $\deg \lambda < \deg \pi$ if $c^{-1}d \not\equiv 0 \pmod{\mathfrak{p}}$ and $\lambda = 0$ if $d \equiv 0 \pmod{\mathfrak{p}}$. Then

$$\begin{pmatrix} a\lambda - b & a \\ c\lambda - d & c \end{pmatrix} \in \Gamma_0(\mathfrak{p})$$

and

$$\begin{pmatrix} a\lambda - b & a \\ c\lambda - d & c \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & \lambda \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Furthermore, since the choice of λ was uniquely determined by the matrix M given, any two elements in the set exhibited above are $\Gamma_0(\mathfrak{p})$-inequivalent. $\square$

With this explicit set of coset representatives, we have the following formula for $\mathrm{Tr}(f)$:

Proposition 6.11. *Let f be a modular form of weight k and type l for $\Gamma_0(\mathfrak{p})$. Then*

$$\mathrm{Tr}(f) = f + \pi^{1-k/2}\left(f|_{k,l}[W_{\mathfrak{p}}]\right)|U_{\mathfrak{p}}.$$

Proof. Write as before

$$f_0(z) \stackrel{\text{def}}{=} f|_{k,l}[W_{\mathfrak{p}}] = \pi^{k/2}(\pi z)^{-k} f\left(\frac{-1}{\pi z}\right).$$

Since $\zeta_{1/\pi} = 1$, we have that

$$\begin{aligned}
f_0|_{k,l}\left[\begin{pmatrix} 1/\pi & \lambda/\pi \\ 0 & 1 \end{pmatrix}\right] &= \left(\frac{1}{\pi}\right)^{k/2} f_0\left(\frac{z+\lambda}{\pi}\right) \\
&= \left(\frac{1}{\pi}\right)^{k/2} \pi^{k/2}(z+\lambda)^{-k} f\left(\frac{-1}{z+\lambda}\right) \\
&= (z+\lambda)^{-k} f\left(\frac{-1}{z+\lambda}\right) \\
&= f|_{k,l}\left[\begin{pmatrix} 0 & -1 \\ 1 & \lambda \end{pmatrix}\right].
\end{aligned}$$

Using the coset representatives from Lemma 6.10 we have thus

$$\begin{aligned}
\mathrm{Tr}(f) &= f + \sum_{\substack{\lambda \in A \\ < d}} f|_{k,l}\left[\begin{pmatrix} 0 & -1 \\ 1 & \lambda \end{pmatrix}\right] \\
&= f + \sum_{\substack{\lambda \in A \\ < d}} f_0|_{k,l}\left[\begin{pmatrix} 1/\pi & \lambda/\pi \\ 0 & 1 \end{pmatrix}\right] \\
&= f + \sum_{\substack{\lambda \in A \\ < d}} \left(\frac{1}{\pi}\right)^{k/2} f_0\left(\frac{z+\lambda}{\pi}\right) \\
&= f + \pi^{1-k/2} f_0|U_{\mathfrak{p}}.
\end{aligned}$$

And the result follows from the definition of f_0. $\square$

6.3 Correspondence

We are now in a position to prove the theorems stated at the beginning of this Chapter.

Theorem 6.2. *Let f be a modular form of weight k and type l for $\Gamma_0(\mathfrak{p})$, with rational u-series coefficients. Then f is a "$\mathfrak{p}$-adic Drinfeld modular form" for $\mathrm{GL}_2(A)$.*

Remark 6.12. *We use here the term "$\mathfrak{p}$-adic Drinfeld modular form" simply to mean a formal u-series $\sum a_j u^j$ such that there exists a sequence $\{f_i\}$ of Drinfeld modular forms on $\mathrm{GL}_2(A)$ such that $v_{\mathfrak{p}}(f_i - f) \to \infty$ as $i \to \infty$. We do not know as yet the extent to which the coefficients of these $\mathfrak{p}$-adic Drinfeld modular forms have nice arithmetic properties.*

Proof. For any positive integer n and g_d the Eisenstein series of weight $q^d - 1$ and type 0 for $\mathrm{GL}_2(A)$, define

$$\begin{aligned} g_{(0)} &\stackrel{\text{def}}{=} (g_d)^n - \pi^{n(q^d-1)/2} (g_d)^n \,|_{n(q^d-1),0}[W_{\mathfrak{p}}] \\ &= (g_d)^n - \pi^{n(q^d-1)} (g_d)^n \,|V_{\mathfrak{p}}. \end{aligned}$$

It is a modular form of weight $n(q^d-1)$ and type 0 for $\Gamma_0(\mathfrak{p})$. Since $(g_d)^n \,|V_{\mathfrak{p}}$ has integral coefficients by Proposition 6.6, we see that $g_{(0)}$ is congruent to 1 modulo $\mathfrak{p}$. Furthermore,

$$\begin{aligned} g_{(0)}|_{n(q^d-1),0}[W_{\mathfrak{p}}] &= (g_d)^n \,|_{n(q^d-1),0}[W_{\mathfrak{p}}] - \pi^{n(q^d-1)/2} (g_d)^n \\ &= \pi^{n(q^d-1)/2} (g_d)^n \,|V_{\mathfrak{p}} - \pi^{(q^d-1)/2} (g_d)^n \\ &= \pi^{n(q^d-1)/2} ((g_d)^n \,|V_{\mathfrak{p}} - (g_d)^n) \\ &\equiv 0 \pmod{\mathfrak{p}^{n(q^d-1)/2+1}}. \end{aligned}$$

The last congruence follows from noticing that $(g_d)^n \,|V_{\mathfrak{p}} = ((g_d)^n - 1) \,|V_{\mathfrak{p}} + 1$ and applying Proposition 6.6 to the u-series $(g_d)^n - 1$, which has valuation at least 1, so that $(g_d)^n|V_{\mathfrak{p}} \equiv (g_d)^n \pmod{\mathfrak{p}}$.

With n fixed as before, define $g_{(r)} = (g_{(0)})^{p^r}$. Since $g_{(0)} \equiv 1 \pmod{\mathfrak{p}}$, we have that $g_{(r)} = (g_{(0)})^{p^r} \equiv 1 \pmod{\mathfrak{p}^{p^r}}$. Similarly, because $g_{(0)}|_{n(q^d-1),0}[W_{\mathfrak{p}}] \equiv 0 \pmod{\mathfrak{p}^{n(q^d-1)/2+1}}$ and $g_{(r)}|_{p^r n(q^d-1),0}[W_{\mathfrak{p}}] = (g_{(0)}|_{n(q^d-1),0}[W_{\mathfrak{p}}])^{p^r} \equiv 0 \pmod{\mathfrak{p}^{np^r(q^d-1)/2+p^r}}$.

The function $fg_{(r)}$ is a Drinfeld modular form of weight $k + np^r(q^d - 1)$ and type l for $\Gamma_0(\mathfrak{p})$ with rational coefficients. Thus by Proposition 6.11, $\mathrm{Tr}(fg_{(r)})$ is of weight $k + np^r(q^d - 1)$ and type l for $\mathrm{GL}_2(A)$ and we have

$$\mathrm{Tr}(fg_{(r)}) - f = (\mathrm{Tr}(fg_{(r)}) - fg_{(r)}) + f(g_{(r)} - 1).$$

We first bound the valuation at $\mathfrak{p}$ of the term $f(g_{(r)} - 1)$ from below, using the fact that $g_{(r)} \equiv 1 \pmod{\mathfrak{p}^{p^r}}$:

$$v_{\mathfrak{p}}(f(g_{(r)} - 1)) \geq p^r + v_{\mathfrak{p}}(f).$$

We consider now

$$\mathrm{Tr}(fg_{(r)}) - fg_{(r)} = \pi^{1-(k+np^r(q^d-1))/2} \left((fg_{(r)})|_{k+np^r(q^d-1),l}[W_{\mathfrak{p}}]\right)|U_{\mathfrak{p}}.$$

Since we have $v_{\mathfrak{p}}(f|U_{\mathfrak{p}}) \geq v_{\mathfrak{p}}(f)$, it follows that:

$$\begin{aligned}
v_{\mathfrak{p}}(\mathrm{Tr}(fg_{(r)}) - fg_{(r)})) &\geq 1 - (k + np^r(q^d - 1))/2 + v_{\mathfrak{p}}\left((fg_{(r)})|_{k+np^r(q^d-1),l}[W_{\mathfrak{p}}]\right) \\
&= 1 - (k + np^r(q^d - 1))/2 + v_{\mathfrak{p}}\left(f|_{k,l}[W_{\mathfrak{p}}]\right) + v_{\mathfrak{p}}\left(g_{(r)}|_{np^r(q^d-1),0}[W_{\mathfrak{p}}]\right) \\
&= 1 - (k + np^r(q^d - 1))/2 + v_{\mathfrak{p}}\left(f|_{k,l}[W_{\mathfrak{p}}]\right) + np^r(q^d - 1)/2 + p^r \\
&= 1 - k/2 + v_{\mathfrak{p}}\left(f|_{k,l}[W_{\mathfrak{p}}]\right) + p^r.
\end{aligned}$$

We conclude that

$$v_{\mathfrak{p}}(\mathrm{Tr}(fg_{(r)}) - f) \geq \inf\{p^r + v_{\mathfrak{p}}(f), p^r + 1 - k/2 + v_{\mathfrak{p}}\left(f|_{k,l}[W_{\mathfrak{p}}]\right)\}.$$

Now since f has rational u-series coefficients, then so does $f|_{k,l}[W_\mathfrak{p}]$, since the Fricke involution of $X_0(\mathfrak{p})$ is defined over the rationals. Thus both $v_\mathfrak{p}(f)$ and $v_\mathfrak{p}\left(f|_{k,l}[W_\mathfrak{p}]\right)$ are finite.

Therefore we have that $v_\mathfrak{p}(\mathrm{Tr}(fg_{(r)}) - f) \to \infty$ as $r \to \infty$, and $\{\mathrm{Tr}(fg_{(r)})\}$ is the sequence of Drinfeld modular forms satisfying the requirements of the definition of a "$\mathfrak{p}$-adic Drinfeld modular form." □

From this follows the second main theorem of this Chapter:

Theorem 6.1. *Let $q \geq 3$. There is a one-to-one correspondence between forms of weight 2 and type 1 for $\Gamma_0(\mathfrak{p})$ with rational $\mathfrak{p}$-integral u-series coefficients and forms of weight $q^d + 1$ and type 1 for $\mathrm{GL}_2(A)$ with rational $\mathfrak{p}$-integral u-series coefficients.*

Proof. We begin by noting that for f modular of weight 2 and type 1 on $\Gamma_0(\mathfrak{p})$, we have that

$$\mathrm{Tr}(f|_{2,1}[W_\mathfrak{p}]) = f|_{2,1}[W_\mathfrak{p}] + f|U_\mathfrak{p}$$

is a modular form of weight 2 and type 1 on $\mathrm{GL}_2(A)$. However, for all $q \neq 2$ this space contains no non-zero modular forms: In any case the full algebra of modular forms for $\mathrm{GL}_2(A)$ is generated by g and h. If $q \geq 4$, then g is of weight $q-1 \geq 3$, and so there are no forms of weight 2. When $q = 3$, the forms of weight 2 are multiples of g, which have type 0, and so there are no non-zero forms of weight 2 and type 1. We conclude that $f|_{2,1}[W_\mathfrak{p}] = -f|U_\mathfrak{p}$. Therefore using Corollary 6.5 and the fact that the Fricke involution is rational, we have that $f|_{2,1}[W_\mathfrak{p}]$ has rational $\mathfrak{p}$-integral u-series coefficients.

When $q = 2$, there are no non-trivial types, and the space of forms of weight 2 for $\mathrm{GL}_2(A)$ is spanned by g^2. Thus we cannot guarantee that $\mathrm{Tr}(f|_{2,1}[W_\mathfrak{p}]) = 0$, or even that $f|_{2,1}[W_\mathfrak{p}]$ has $\mathfrak{p}$-integral coefficients. In the classical case the forms on $\Gamma_0(\ell)$ such that

$\mathrm{Tr}(f) = \mathrm{Tr}(f|[W_\ell]) = 0$ are exactly linear combinations of newforms. It is reasonable to conjecture that a similar result holds here and that the existence of oldforms of weight 2 for $\Gamma_0(\mathfrak{p})$ is exactly the obstruction to the result we seek.

Going back to the case $q \geq 3$, writing $g_{(0)} = g_d - \pi^{(q^d-1)/2} g_d|_{q^d-1,0}[W_\mathfrak{p}]$ as before, we consider the map $f \to \mathrm{Tr}(fg_{(0)})$. This map takes an element of the space of forms of weight 2 and type 1 for $\Gamma_0(\mathfrak{p})$ to a form of weight q^d+1 and type 1 for $\mathrm{GL}_2(A)$. We have that g_d has integral coefficients, and since it is a form for $\mathrm{GL}_2(A)$ so does $g_d|_{q^d-1,0}[W_\mathfrak{p}]$ by Lemma 6.7. Thus $g_{(0)}$ and $g_{(0)}|_{q^d-1,0}[W_\mathfrak{p}]$ have integral u-series coefficients. Now from the formula

$$\mathrm{Tr}(fg_{(0)}) = fg_{(0)} + \left(f|_{2,1}[W_\mathfrak{p}]g_{(0)}|_{q^d-1,0}[W_\mathfrak{p}]\right)|U_\mathfrak{p}.$$

we conclude that $\mathrm{Tr}(fg_{(0)})$ also has rational, $\mathfrak{p}$-integral u-series coefficients. Thus the map $f \to \mathrm{Tr}(fg_{(0)})$ preserves rationality and $\mathfrak{p}$-integrality of the u-series expansion coefficients.

From the computations in the proof of Theorem 6.2, we have

$$v_\mathfrak{p}(\mathrm{Tr}(fg_{(0)}) - f) \geq \inf\{1 + v_\mathfrak{p}(f), 1 + v_\mathfrak{p}\left(f|_{2,1}[W_\mathfrak{p}]\right)\} \geq 1,$$

so that $f \equiv \mathrm{Tr}(fg_{(0)}) \pmod{\mathfrak{p}}$.

Now consider N the set of $\tilde{f} \in \mathbb{F}_\mathfrak{p}[[u]]$ such that there is f of weight 2 and type 1 for $\Gamma_0(\mathfrak{p})$ with rational, $\mathfrak{p}$-integral coefficients with $f \equiv \tilde{f} \pmod{\mathfrak{p}}$. As mentioned in Chapter 3, the space of forms of weight 2 and type 1 for $\Gamma_0(\mathfrak{p})$ is of dimension $g_\mathfrak{p}+1$ and has a basis of forms with integral u-series coefficients, and from this it follows that N has dimension $g_\mathfrak{p}+1$ as an $\mathbb{F}_\mathfrak{p}$-vector space.

Since $f \equiv \mathrm{Tr}(fg_{(0)}) \pmod{\mathfrak{p}}$, N is a subspace of the $\mathbb{F}_\mathfrak{p}$-vector space $\tilde{M}_{q^d+1,1}$, the space that contains the reductions modulo $\mathfrak{p}$ of all of the forms of weight q^d+1 and type

1 for $\mathrm{GL}_2(A)$ with rational, $\mathfrak{p}$-integral u-series coefficients. However, the space $\tilde{M}_{q^d+1,1}$ also has dimension $g_{\mathfrak{p}} + 1$, since $M_{q^d+1,1}(\mathrm{GL}_2(A))$ has a basis of forms with integral u-series coefficients. Thus $N = \tilde{M}_{q^d+1,1}$, and the trace map establishes a one-to-one correspondence between the spaces, as claimed. $\square$

Chapter 7

Weierstrass points

In this Chapter we will investigate the Weierstrass points of the curve $X_0(\mathfrak{p})$, for $\mathfrak{p}$ as always a prime ideal generated by a monic polynomial $\pi(T)$ of degree d.

7.1 Weierstrass points in characteristic p

Since the theory of Weierstrass points in characteristic p is much less well known than the theory in characteristic 0, we begin with a short review of the facts we will need, based on the treatment in [48] and [25]. In particular, proofs of all facts that are stated without proof here can be found in [25].

For the duration of this section and the next, let k be an algebraically closed field and X a smooth complete projective irreducible curve over k of genus $g \geq 2$ with function field $k(X)$. A natural question to ask about X is the following: For P a point of X and n a positive integer, does there exist a function F on X such that F has a pole of order exactly n at P and F is regular elsewhere? If the answer to this question is negative, we say that n is a *gap* at P; otherwise n is a *pole number* at P. It is a fact that there are exactly g gaps at P, and if $n_1, \ldots, n_g$ are the gaps at P, indexed such that $n_i < n_j$ if $i < j$, we say that $(n_1, \ldots, n_g)$ is the gap sequence at P.

For a fixed curve X, it can be shown that there exists a sequence of positive integers

$(n_1, \ldots, n_g)$ with $n_i < n_j$ if $i < j$ such that $(n_1, \ldots, n_g)$ is the gap sequence for all but finitely many points of X. We call this sequence the *canonical gap sequence* of X. The finitely many points that have a different gap sequence are called the *Weierstrass points* of X. If $(n_1, \ldots, n_g)$ is the canonical gap sequence of X and $(n'_1, \ldots, n'_g)$ is the gap sequence at P for any point P of X, then $n_i \leq n'_i$ for each i.

For any point P on X, we define its Weierstrass weight to be:

$$\mathrm{wt}(P) = \sum_{i=1}^{g} n'_i - n_i.$$

Thus a point is a Weierstrass point if and only if it has positive Weierstrass weight.

Remark 7.1. *This is not the usual definition of the Weierstrass weight; the literature uses the quantity $v_P(w(\phi, s))$ which we will define in the next section. Often (an important example is when k is of characteristic 0) the two definitions agree. When they do not, we feel that the quantity that we define as the Weierstrass weight, while possibly harder to compute, is much more natural.*

If X is defined over a field of characteristic 0, then the canonical gap sequence is always $(1, \ldots, g)$. When k is of characteristic $p > 0$ and X has canonical gap sequence $(1, \ldots, g)$, we say that X has a *classical* canonical gap sequence, or a classical canonical linear system (this designation will be justified shortly when we define the canonical orders of X).

Example 7.2. *Let X be a hyperelliptic curve of genus $g \geq 2$, then its canonical gap sequence is $(1, \ldots, g)$. (In characteristic $p > 0$ this is a theorem that was implicit in [30] and stated explicitly in the seminal work of Schmidt [44] defining Weierstrass points in positive characteristic.) Furthermore, the Weierstrass points of X are exactly the* *branch points of f, where $f\colon X \to \mathbb{P}^1$ is any degree 2 morphism. At such a branch point P the function $F = \frac{1}{f-f(P)}$ has a double pole at P and is regular elsewhere, and so at the Weierstrass points the gap sequence is $(1, 3, \ldots, 2g-1)$.*

Example 7.3. *The projective curve of genus 3 given by $X_0^4 + X_1^4 + X_2^4 = 0$ over $\bar{\mathbb{F}}_3$ does not have a classical gap sequence. On this curve, for each point P one can construct a function having a pole of order ≤ 3 at P and regular elsewhere.*

In practice, it is often more convenient to consider a related sequence of strictly increasing positive integers $(j_1, \ldots, j_{g-1})$ called the *canonical orders* of X, which we now describe. For any element $x \in k(X)$, we will write $[x]$ for the divisor of x, $\sum_P v_P(x)P$, where the sum is taken over all points P of X. As usual, for any divisor D on X, we may define the linear system

$$L(D) = \{x \in k(X)^* : [x] \geq -D\} \cup \{0\}.$$

We further denote by Ω_X the space of (algebraic) meromorphic differential forms on X. Because X is defined over an algebraically closed field, we have a canonical isomorphism between Ω_X and the space of Weil differentials W_X (in fact, to obtain this isomorphism it would suffice here to require that $k' \otimes k(X)$ be a field for all finite extensions k' of k). This allows us to define the divisor $[\omega]$ for ω a meromorphic differential on X. We do this in the following manner: Let $\mathbb{A}_{k(X)}$ denote the ring of adèles of $k(X)$ and for D a divisor on X, write

$$\mathbb{A}_{k(X)}(D) \stackrel{\text{def}}{=} \{\alpha = (\alpha_P) \in \mathbb{A}_{k(X)} \mid v_P(\alpha_P) \geq -v_P(D) \text{ for all points } P \text{ of } X\}.$$

Then a Weil differential on X is a k-linear functional with domain $\mathbb{A}_{k(X)}$ that vanishes on $\mathbb{A}_{k(X)}(D) + k(X)$ for some divisor D. For each Weil differential ω^*, there is a unique divisor D of maximum degree such that ω^* vanishes on $\mathbb{A}_{k(X)}(D) + k(X)$, and we define $[\omega^*] \stackrel{\text{def}}{=} D$. Then if ω corresponds to ω^* under the canonical isomorphism between Weil differentials and meromorphic differentials, we simply write $[\omega] = [\omega^*]$ and $v_P(\omega) = v_P([\omega])$. One pleasant consequence of this definition is that for $x \in k(X)$ and $\omega \in \Omega_X$ we have $[x\omega] = [x] + [\omega]$. If ω is a meromorphic differential on X, its divisor C is called a *canonical divisor* on X, and since any two meromorphic differentials differ by a function, any two canonical divisors are linearly equivalent.

For a point P of X, consider the following sequence of spaces:

$$k = L(0) \subseteq L(P) \subseteq L(2P) \subseteq L(3P) \subseteq \ldots$$

Then we have that n is a gap at P if and only if $L((n-1)P) = L(nP)$. By the Riemann-Roch theorem, we have that for any positive integer n and any point P,

$$\dim L(nP) = n - g + 1 + \dim L(C - nP),$$

from which it follows that

$$\dim L((n+1)P)/L(nP) = 1 - \dim L(C - nP)/L(C - (n+1)P).$$

Writing $L_C(nP) = L(C - nP)$, this last equation justifies our interest in the *(canonical) osculating filtration* at P:

$$L(C) = L_C(0) \supseteq L_C(P) \supseteq L_C(2P) \supseteq L_C(3P) \supseteq \ldots$$

Indeed, for a positive integer n, $n+1$ is a gap at P if and only if $L_C(nP) \supsetneq L_C((n+1)P)$. In turn, this implies the existence of a function $F \in L(C)$ such that $v_P(F) = n - v_P(C)$. Whenever such a function exists, we say that n is a *canonical order* at P; the existence of such a function does not depend on the choice of canonical divisor C. Thus for a positive integer n, n is a canonical order at P if and only if $n+1$ is a gap at P. (We note that since X is a curve over an algebraically closed field, the existence of a point P such that 1 is a pole number at P implies that X has genus zero, so that in our case 1 will always be a gap for any point P on X.) As was the case for gap sequences, all but finitely many points of X have the same canonical orders, and we call the strictly increasing sequence of positive integers $(j_1, \ldots, j_{g-1})$ formed by these integers the *canonical orders* of X.

If $(j_1, \ldots, j_{g-1})$ are the canonical orders of X and $(j'_1, \ldots, j'_{g-1})$ are the canonical orders at P for any point P of X, then again $j_i \leq j'_i$ for each i, and the Weierstrass weight of P can alternatively be computed as:

$$\mathrm{wt}(P) = \sum_{i=1}^{g-1} j'_i - j_i.$$

The point P is called an *osculation point* of X if $j'_{g-1} > g-1$. In particular an osculation point has at least one pole number that is less than or equal to g. If X has a classical gap sequence, then the osculation points and the Weierstrass points of X exactly coincide. Otherwise, every point of X is an osculation point. This simple observation allows us, in joint work with Armana, to show that $X_0(\mathfrak{p})$, for $\mathfrak{p}$ generated by an ideal of degree at least 3, has a classical gap sequence. We note that our argument corrects a small inaccuracy in Armana's original work.

Proposition 7.4. *Let $\mathfrak{p}$ be a prime ideal of degree at least 3 in $\mathbb{F}_q[T]$. Then $X_0(\mathfrak{p})$ has a classical gap sequence.*

Proof. Using an argument analogous to Ogg's argument in the classical case, Armana shows the following [4]: Let P be a $\mathbb{F}_q(T)$-rational point of $X_0(\mathfrak{p})$ such that its unique extension to a section of $M_0(\mathfrak{p})$ ($M_0(\mathfrak{p})$ is the moduli scheme associated to $X_0(\mathfrak{p})$, see Chapter 3) over $\mathbb{F}_q[T]$ is not supersingular at $\mathfrak{p}$, and denote by $c \geq 1$ be the smallest pole number at P. Then $c \geq 1 + g_{\mathfrak{p}}$, where as before $g_{\mathfrak{p}}$ is the genus of $X_0(\mathfrak{p})$. Thus such a point is not an osculation point of the curve.

Either one of the cusps of $X_0(\mathfrak{p})$ satisfies the conditions on the point P above. Thus $X_0(\mathfrak{p})$ has a point that is not an osculation point, and the result follows. □

This argument further implies that the $\mathbb{F}_q(T)$-rational Weierstrass points of $X_0(\mathfrak{p})$ have supersingular reduction modulo $\mathfrak{p}$. This result can also be deduced in a completely different way by using the following theorem proved by Baker [6]:

Theorem 7.5. *Let R be a complete discrete valuation ring with algebraically closed residue field k. Let X be a smooth, proper, geometrically connected curve defined over the fraction field of R, and denote by $\mathfrak{X}$ a proper model for X over R. (In other words, $\mathfrak{X}$ is a proper flat scheme over* $\operatorname{Spec} R$ *such that its generic fiber is X.) Suppose that the special fiber of $\mathfrak{X}$ consists of two genus* 0 *curves intersecting transversally at* 3 *or more points. Then every Weierstrass point of X defined over the fraction field of R specializes to a singular point of the special fiber of $\mathfrak{X}$.*

The proof of this Theorem is a corollary of a Specialization Lemma proved in the same paper, which roughly says that the dimension of a linear system can only increase under specialization from the curve X to the dual graph of the model $\mathfrak{X}$.

It now follows easily that the $K_{\mathfrak{p}}^{\mathrm{un}}$-rational Weierstrass points of $X_0(\mathfrak{p})$ have supersingular reduction modulo $\mathfrak{p}$, where $K_{\mathfrak{p}}$ is the completion of $K = \mathbb{F}_q(T)$ at $\mathfrak{p}$ and $K_{\mathfrak{p}}^{\mathrm{un}}$ is its maximal unramified extension: As remarked in Section 3.2.2, for $X_0(\mathfrak{p})$ there exists a proper model $\bar{M}_0(\mathfrak{p})$ over $R_{\mathfrak{p}}$, where $R_{\mathfrak{p}}$ is the ring of integers of $K_{\mathfrak{p}}^{\mathrm{un}}$. Furthermore, the special fiber of $\bar{M}_0(\mathfrak{p})$ consists of two genus 0 curves intersecting transversally at the supersingular points, of which there are at least 3 if $d \geq 3$ and $q \geq 3$ or if $d \geq 4$

and $q = 2$. Thus $X_0(\mathfrak{p})$ satisfies the conditions of Theorem 7.5. We conclude that the Weierstrass points of $X_0(\mathfrak{p})$ that are defined over $K_{\mathfrak{p}}^{\mathrm{un}}$ reduce modulo $\mathfrak{p}$ to the singular points of the special fiber. But those are exactly the supersingular points.

7.2 The Wronskian function

It is natural to ask whether it is possible to say more about the connection between the supersingular locus at $\mathfrak{p}$ and the Weierstrass points of $X_0(\mathfrak{p})$, as was done in the classical case by Rohrlich [43], and Ahlgren and Ono [2]. An important tool to answer this question is a function W on X, the Wronskian, such that $v_P(W) = 0$ if P is not a Weierstrass point of X and such that $v_P(W) \geq \mathrm{wt}(P)$, whose construction is due to Stöhr and Voloch [48].

A *separating variable* for $k(X)$ is an element $s \in k(X)$ transcendental over k such that $k(X)$ is a finite, separable extension of $k(s)$. With the assumptions on X stated in the previous Section, we have that s is a separating variable if and only if the differential ds is not identically 0. Furthermore, s is a separating variable if s is a local parameter at a separable point of X. Since in our case X is defined over an algebraically closed field k, every point is separable.

On the polynomial ring $k[s]$, we may define the n^{th} *Hasse derivative with respect to* s by putting

$$\mathfrak{D}_s^{(n)}(s^m) = \begin{cases} \binom{m}{n} s^{m-n} & \text{if } m \geq n, \\ 0 & \text{otherwise,} \end{cases}$$

and extending linearly to $k[s]$. It can be shown that if s is a separating variable for $k(X)$ over k, then this family of maps can be uniquely extended to a family of maps

$\mathfrak{D}_s^{(n)} : k(X) \to k(X)$.

Again let C be a canonical divisor on the curve X, and consider the linear system $L(C)$ associated to it. It is a basic fact that $L(C)$ is a k-vector subspace of $k(X)$ of dimension g, and that replacing C by a different canonical divisor yields an isomorphic subspace. Fix any basis $\phi = \{\phi_1, \ldots \phi_g\}$ of $L(C)$, and define the matrix

$$H = H(\phi, s) = \left(\mathfrak{D}_s^{(j)}(\phi_i)\right)$$

for $1 \leq i \leq g$ and $0 \leq j$. Write further $H^{(j)}$ for the column of H whose i^{th} entry is $\mathfrak{D}_s^{(j)}(\phi_i)$.

We are interested in the indices j such that $H^{(j)}$ is not a $k(X)$-linear combination of lower numbered columns. This is true for $j = 0$ since the ϕ_i's are not all zero. It is not hard to show that there are $g - 1$ more such indices, which we will denote by $j_1, \ldots, j_{g-1}$, and we will write $J(\phi, s) = (j_1, \ldots, j_{g-1})$. It can be shown that $J(\phi, s)$ in fact does not depend on our choice of s a separating variable, C a canonical divisor or ϕ a basis for the linear system associated to C, and in fact that the j_i's are exactly the canonical orders of X defined earlier.

For any sequence $J = (j_1, j_2, \ldots)$ of positive integers, let H^J be the submatrix of H whose first column is $H^{(0)}$ and whose $(l + 1)^{st}$ column is $H^{(j_l)}$. Then we may define

$$W(\phi, s) = \det H^{J(\phi,s)},$$

the *Wronskian of ϕ with respect to s.* While not independent of the choices made above, this function behaves as well as well as can be expected. More precisely, put $\phi'_i = \sum_j a_{ij}\phi_j$ for $a_{ij} \in k$ such that $\phi' = (\phi'_1, \ldots, \phi'_g)$ is a different basis for $L(C)$, and let $y \in k(X)^*$ and t be another separating variable. Then

$$W(y\phi', t) = \det(a_{ij})y^g\,(ds/dt)^{j_1+\ldots+j_{g-1}}\,W(\phi, s). \tag{7.1}$$

In light of this equation, we define the following divisor:

$$w(\phi, s) = [W(\phi, s)] + gC + (j_1 + \ldots + j_{g-1})[ds],$$

which by equation (7.1) is in fact independent of any choice we made. One can show that the points in the support of $w(\phi, s)$ are exactly the Weierstrass points of X.

The divisor $w(\phi, s)$ is effective: Fixing a point P of X, one may choose a canonical divisor C such that $v_P(C) = 0$, which ensures that $v_P(\phi_i) \geq 0$, so that $v_p([W(\phi, s)]) \geq 0$ since taking Hasse derivatives does not lower the valuation. Furthermore, one can choose s to be a local parameter at P, so that $v_P([ds]) = 0$. With these choices and because of the invariance of $w(\phi, s)$, it follows that $v_P(w(\phi, s)) \geq 0$ for each P. In addition, the Wronskian W constructed with this choice of s has the property that $v_P(W) = 0$ if P is not a Weierstrass point of X, and that $v_P(W) = v_P(w(\phi, s)) \geq \mathrm{wt}(P)$.

Remark 7.6. *If X is defined over a field of characteristic 0, we have the equality $v_P(w(\phi, s)) = \mathrm{wt}(P)$. In finite characteristic, this equality holds if and only if $\det \binom{J'}{J} \neq 0$, where $J' = (j'_1, \ldots, j'_{g-1})$ is the sequence of canonical orders at P, $J = (j_1, \ldots, j_{g-1})$ is the sequence of canonical orders of X, and $\binom{J'}{J}$ is the $(g-1) \times (g-1)$ matrix of binomial coefficients $\binom{j'_r}{j_s}$, where $\binom{j'_r}{j_s} = 0$ if $j'_r < j_s$ and each binomial coefficient is reduced modulo p, the characteristic of k.*

We will also need the following well-known fact: We have that $\dim_{k(X)} \Omega_X = 1$, so that $\Omega_X = k(X) \cdot \omega$ for any non-zero $\omega \in \Omega_X$. If C is a canonical divisor of X, by definition it is the divisor of some Weil differential ω^* and thus of a meromorphic differential ω. Then the map

$$\Omega_X \to k(X)$$

$$x\omega \mapsto x$$

is an isomorphism of k-vector spaces, and under this isomorphism the space $L(C) \subset k(X)$ corresponds to the space $\Omega_{X,\text{alg}}$ of algebraic regular differentials.

7.3 The modular Wronskian

To refine the connection between Weierstrass points on $X_0(\mathfrak{p})$ and the supersingular locus in characteristic $\mathfrak{p}$, we now specialize the ideas of the previous two sections to the curve $X_0(\mathfrak{p})$ over C, as Rohrlich did in the classical setting. We will consider the rigid analytic structure on $X_0(\mathfrak{p})$, as it is particularly amenable to computations thanks to the rich theory of Drinfeld modular forms. For ease of reading, we will write “analytic” below to mean rigid analytic. An analytic function without poles will be a holomorphic function, and an analytic function possibly with poles will be said to be meromorphic.

We first note that GAGA theorems hold for rigid analytic geometry [34], [35]. More precisely, we will need the following: Let X be a smooth projective algebraic curve defined over a complete non-archimedean field k of finite characteristic p, and let X^{an} be the rigid analytic space associated to it (see for example [15] for the construction of X^{an}). We note that the sets of points of X and X^{an} coincide, so that we will not make a distinction between a divisor on X and a divisor on X^{an}. We denote by O the sheaf of algebraic regular functions on X, and by $\mathcal{O}$ the sheaf of holomorphic functions on X^{an}. Then there is an equivalence of category between the category of algebraic coherent sheaves on X and analytic coherent sheaves on X^{an}. In particular, to every algebraic coherent sheaf F on X one may associate an analytic coherent sheaf F^{an} on X^{an}.

The linear space $L(D)$ associated to a divisor D on X is in fact the space of global sections of an algebraic sheaf which we will also denote by $L(D)$. $L(D)$ is a subsheaf of the sheaf of rational functions M on X such that for S a Zariski open set of X,

$$L(D)(S) = \{f \in M(S) \mid [f] \geq -D|_S\} \cup \{0\},$$

where $-D|_S$ is the restriction of the divisor $-D$ to the set S. The sheaf $L(D)$ is coherent and thus corresponds to a sheaf $L(D)^{an}$ on X^{an}.

Because the operation $*^{an}$ commutes with duals and tensor products, $L(D)^{an}$ is none other than $\mathcal{L}(D)$, the subsheaf of meromorphic functions $\mathcal{M}$ on X^{an} such that for U an open set of X^{an} we have

$$\mathcal{L}(D)(U) = \{f \in \mathcal{M}(U) \mid [f] \geq -D|_U\} \cup \{0\}.$$

In particular, as a consequence of GAGA, the space of global sections of $L(D)$ is isomorphic to the space of global sections of $\mathcal{L}(D)$, and for a point P of X^{an} we may instead consider the sequence of spaces

$$k = \mathcal{L}(0)(X^{an}) \subseteq \mathcal{L}(P)(X^{an}) \subseteq \mathcal{L}(2P)(X^{an}) \subseteq \mathcal{L}(3P)(X^{an}) \subseteq \dots$$

By GAGA $L((n-1)P)(X) = L(nP)(X)$ if and only if $\mathcal{L}((n-1)P)(X^{an}) = \mathcal{L}(nP)(X^{an})$, so that the gap sequence and the Weierstrass weight can be computed analytically.

Arguing as in the algebraic case, we obtain that if j' is a canonical order at P, there is $F \in \mathcal{L}(C)(X^{an})$ such that $v_P(F) = j' - v_P(C)$.

We now start our work on $X_0(\mathfrak{p})$ in earnest. Throughout, when we say that P is a point on $X_0(\mathfrak{p})$, we will always mean a C-valued point. We first work out the divisor of dz: We have that $\frac{1}{u^2}du = -\tilde{\pi}dz$, but as discussed in Section 3.1.4, u is not actually an analytic parameter at ∞, $t = u^{q-1}$ is. Substituting, we have $\frac{1}{t^{q/(q-1)}}dt = \tilde{\pi}dz$, and dz has a pole of order $\left(1 + \frac{1}{q-1}\right)$ at ∞. Similarly, if P is not a cusp of $X_0(\mathfrak{p})$, choose $\tau \in \Omega$ to be a representative of P in the Drinfeld upper half-plane, and write e for the order of the stabilizer of τ in $\widetilde{\Gamma_0}(\mathfrak{p}) = \Gamma_0(\mathfrak{p})/\Gamma_0(\mathfrak{p}) \cap Z(\mathrm{GL}_2(A))$. Then we may choose $t = (z-\tau)^e$ as an analytic parameter at P. We have $dz = \frac{1}{e}t^{(e-1)/e}dt$ (e is a divisor of $q+1$ so it is prime to the characteristic p of C [18]) and so dz has a pole of order $\left(1 - \frac{1}{e}\right)$ at τ.

Proposition 7.7. *Let P be a point on $X_0(\mathfrak{p})$, and write $j'_0 = 0$, and $(j'_1, \ldots, j'_{g_\mathfrak{p}-1})$ for the canonical orders at P. If P is not a cusp, choose $\tau \in \Omega$ to be a representative of P in the Drinfeld upper-half plane, and write e for the order of the stabilizer of τ in $\widetilde{\Gamma_0}(\mathfrak{p})$. If P is a cusp we write $\tau = 0$ or $\tau = \infty$. Then there is a basis $\{f_i\}_{i=0}^{g_\mathfrak{p}-1}$ of $M^2_{2,1}(\Gamma_0(\mathfrak{p}))$ such that:*

$$\mathrm{ord}_\tau(f_i) = \begin{cases} ej'_i + e - 1 & \text{if } \tau \in \Omega, \\ (q-1)j'_i + q & \text{if } \tau = 0, \infty. \end{cases}$$

Proof. Fix a point P on $X_0(\mathfrak{p})$, and let s be a parameter at P. We choose as our canonical divisor the divisor $[ds]$. Let $(j'_1, \ldots, j'_{g_\mathfrak{p}-1})$ be the canonical orders at P. Then there is a basis $\{F_0, \ldots, F_{g_\mathfrak{p}-1}\}$ of $\mathcal{L}([ds])$ such that $\mathrm{ord}_P(F_i) = j'_i$. Furthermore, $\{F_i ds\}$ is a basis for the space of regular differentials on $X_0(\mathfrak{p})$. Because of the correspondence between the space $M^2_{2,1}(\Gamma_0(\mathfrak{p}))$ of double cusp forms of weight 2 and type 1 for $\Gamma_0(\mathfrak{p})$ and the space of regular differentials on $X_0(\mathfrak{p})$, we have that there is a basis $\{f_i\}$ for $M^2_{2,1}(\Gamma_0(\mathfrak{p}))$ such that $f_i(z)dz = F_i ds$. In particular, $\mathrm{ord}_P(f_i(z)dz) = \mathrm{ord}_P(F_i ds) = \mathrm{ord}_P(F_i) = j'_i$.

Suppose first that P is not a cusp, and choose $\tau \in \Omega$ to be a representative of P in the Drinfeld upper half-plane. Write e for the order of the stabilizer of τ in $\widetilde{\Gamma_0}(\mathfrak{p})$. Then $\mathrm{ord}_P(f_i(z)dz) = \frac{\mathrm{ord}_\tau(f_i)}{e} - \frac{e-1}{e}$. If P is a cusp, then $\mathrm{ord}_P(f_i(z)dz) = \frac{\mathrm{ord}_\infty(f_i(z))}{q-1} - \frac{q}{q-1}$. The result follows. □

For any basis $\{f_0, f_1, \dots f_{g_\mathfrak{p}-1}\}$ of $M^2_{2,1}(\Gamma_0(\mathfrak{p}))$, we are interested in the quantity

$$W(f_0, \dots, f_{g_\mathfrak{p}-1}) = \begin{vmatrix} f_0(z) & D_1(f_0(z)) & \dots & D_{g_\mathfrak{p}-1}(f_0(z)) \\ \vdots & & \vdots & \\ f_{g_\mathfrak{p}-1}(z) & D_1(f_{g_\mathfrak{p}-1}(z)) & \dots & D_{g_\mathfrak{p}-1}(f_{g_\mathfrak{p}-1}(z)) \end{vmatrix},$$

where D_n is the normalized Hasse derivative introduced in Chapter 4. This is a modular form of weight $g_\mathfrak{p}(g_\mathfrak{p}+1)$ for $\Gamma_0(\mathfrak{p})$. We denote by $W(z)$ the normalization of this form that has 1 as its leading coefficient for the u-series expansion at ∞ and call it the modular Wronskian on $X_0(\mathfrak{p})$. We note that if $\{f_0, \dots f_{g_\mathfrak{p}-1}\}$ and $\{f'_0, \dots f'_{g_\mathfrak{p}-1}\}$ are two bases for $M^2_{2,1}(\Gamma_0(\mathfrak{p}))$, then $W(f_0, \dots, f_{g_\mathfrak{p}-1}) = aW(f'_0, \dots, f'_{g_\mathfrak{p}-1})$ for $0 \neq a \in C$.

We are interested in $W(z)$ because of its relation to the Weierstrass points of $X_0(\mathfrak{p})$:

Proposition 7.8. *For any point P of $X_0(\mathfrak{p})$, we have*

$$\mathrm{ord}_P(W(z)(dz)^{g_\mathfrak{p}(g_\mathfrak{p}+1)/2}) \geq \mathrm{wt}(P).$$

In addition, when P is not elliptic and is not a Weierstrass point, we have equality: $\mathrm{ord}_P(W(z)(dz)^{g_\mathfrak{p}(g_\mathfrak{p}+1)/2}) = 0$.

Proof. Let P be a point on $X_0(\mathfrak{p})$, and choose a basis $\{f_i\}$ that satisfies the conclusion of Proposition 7.7. Then

$$\mathrm{ord}_P(W(f_0, \dots, f_{g_\mathfrak{p}-1})(dz)^{g_\mathfrak{p}(g_\mathfrak{p}+1)/2}) = \mathrm{ord}_P(W(z)(dz)^{g_\mathfrak{p}(g_\mathfrak{p}+1)/2}),$$

so we may work with $W(f_0, \ldots, f_{g_\mathfrak{p}-1})$ for convenience.

First, suppose that P is not a cusp of $X_0(\mathfrak{p})$, and choose $\tau \in \Omega$ to be a representative of P in the Drinfeld upper half-plane. By Proposition 4.9, for $k = 0, \ldots, g_\mathfrak{p} - 1$, we have that

$$\mathrm{ord}_\tau(D_k(f_l)) \geq ej'_l + e - 1 - k$$

with equality if and only if $\binom{ej'_l+e-1}{k} \not\equiv 0 \pmod{p}$. When computing the determinant $W(f_0, \ldots, f_{g_\mathfrak{p}-1})$, we will be adding terms all of whose order of vanishing at τ is $\geq \sum_{i=0}^{g_\mathfrak{p}-1}(ej'_i - i + e - 1)$. Thus

$$\mathrm{ord}_\tau W(f_0, \ldots, f_{g_\mathfrak{p}-1}) \geq \sum_{i=0}^{g_\mathfrak{p}-1}(ej'_i - i + e - 1).$$

We have

$$\sum_{i=0}^{g_\mathfrak{p}-1}(ej'_i - i + e - 1) = e\,\mathrm{wt}(P) + \frac{g_\mathfrak{p}(g_\mathfrak{p}+1)}{2}(e-1).$$

Thus

$$\begin{aligned}\mathrm{ord}_P(W(f_0, \ldots, f_{g_\mathfrak{p}-1})(dz)^{g_\mathfrak{p}(g_\mathfrak{p}+1)/2}) &\geq \mathrm{wt}(P) + \frac{g_\mathfrak{p}(g_\mathfrak{p}+1)}{2}\frac{e-1}{e} - \frac{g_\mathfrak{p}(g_\mathfrak{p}+1)}{2}\frac{e-1}{e} \\ &= \mathrm{wt}(P).\end{aligned}$$

.

In the case where P is not elliptic and P is not a Weierstrass point, the terms on the diagonal of $W(f_0, \ldots, f_{g_\mathfrak{p}-1})$ have order of vanishing 0, and all of the terms below the diagonal have order of vanishing strictly greater than 0. Thus $\mathrm{ord}_\tau W(f_0, \ldots, f_{g_\mathfrak{p}-1}) = 0 = \mathrm{wt}(P)$.

Now suppose that P is a cusp of $X_0(\mathfrak{p})$, and we write $\tau = 0$ or $\tau = \infty$. By Proposition 4.10, for $k = 0, \ldots, g_\mathfrak{p} - 1$, we have that

$$\mathrm{ord}_\tau(D_k(f_l)) \geq (q-1)j'_l + q + 1$$

with equality if and only if $\operatorname{ord}_\tau(f_l) \not\equiv 0 \pmod p$. (Here $\operatorname{ord}_\tau(f_l) \geq 2$.) Again, when computing the determinant $W(f_0, \ldots, f_{g_\mathfrak{p}-1})$, we will be adding terms all of whose order of vanishing at τ is $\geq \sum_{i=0}^{g_\mathfrak{p}-1}((q-1)j_i' + q + 1)$. Thus

$$\operatorname{ord}_\tau W(f_0, \ldots, f_{g_\mathfrak{p}-1}) \geq \sum_{i=0}^{g_\mathfrak{p}-1}((q-1)j_i' + q + 1).$$

We have

$$\begin{aligned} \sum_{i=0}^{g_\mathfrak{p}-1}((q-1)j_i' + q + 1) &= (q-1)\operatorname{wt}(P) + g_\mathfrak{p} q + \frac{g_\mathfrak{p}(g_\mathfrak{p}-1)}{2} + (q-1)\frac{g_\mathfrak{p}(g_\mathfrak{p}-1)}{2} \\ &= (q-1)\operatorname{wt}(P) + g_\mathfrak{p} q + q\frac{g_\mathfrak{p}(g_\mathfrak{p}-1)}{2} \\ &= (q-1)\operatorname{wt}(P) + q\frac{g_\mathfrak{p}(g_\mathfrak{p}+1)}{2}. \end{aligned}$$

And so

$$\begin{aligned} \operatorname{ord}_P(W(f_0, \ldots, f_{g_\mathfrak{p}-1})(dz)^{g_\mathfrak{p}(g_\mathfrak{p}+1)/2}) &\geq \operatorname{wt}(P) + \frac{q}{q-1}\frac{g_\mathfrak{p}(g_\mathfrak{p}+1)}{2} - \frac{g_\mathfrak{p}(g_\mathfrak{p}+1)}{2}\frac{q}{q-1} \\ &= \operatorname{wt}(P). \end{aligned}$$

□

Remark 7.9. *Ideally, one would like to show equality for all points P that are not Weierstrass points.*

We will also need the following:

Proposition 7.10. *The u-series coefficients of $W(z)$ at ∞ are rational.*

Proof. We will prove a slightly stronger statement, which we will need later. As remarked before, there is a basis $\{f_1, \ldots, f_{g_\mathfrak{p}}\}$ for the space $M^2_{2,1}(\Gamma_0(\mathfrak{p}))$ that has integral u-series coefficients at ∞.

When computing $W(f_1, \ldots, f_{g_\mathfrak{p}})$, we will compute D_n for $n \leq g_\mathfrak{p} - 1$. From the explicit formula (2.13), we have easily that $g_\mathfrak{p} \leq 2q^{d-2}$, so that $n \leq 2q^{d-2} - 1 < q^d$. In this case, Proposition 4.7 says that D_n preserves $\mathfrak{p}$-integrality of the u-series coefficients, so $W(f_1, \ldots, f_{g_\mathfrak{p}})$ has rational, $\mathfrak{p}$-integral u-series coefficients.

For $a \in K$ the leading coefficient of $W(f_1, \ldots, f_{g_\mathfrak{p}})$, we have that

$$W(z) = \frac{1}{a} W(f_1, \ldots, f_{g_\mathfrak{p}}),$$

and so $W(z)$ has rational u-series coefficients at ∞. $\square$

7.4 A computation

Because of its significance, it would be of great interest to compute the form $W(z)$ explicitly, or in fact to compute its divisor. This task, however, seems extremely difficult. In light of our goal of refining the connection between Weierstrass points and the supersingular locus, an easier but important goal would be to compute its divisor modulo $\mathfrak{p}$. This also seems out of reach at the moment, although some progress can be made in a very specific case, which we describe here.

We will need some notation: For a system of derivatives $\{\delta_n\}$ which is a higher derivation, we will write $W_\delta(f_1, \ldots, f_g)$ for the quantity

$$\begin{vmatrix} f_1 & \delta_1(f_1) & \ldots & \delta_{g-1}(f_1) \\ \vdots & & \vdots & \\ f_g & \delta_1(f_g) & \ldots & \delta_{g-1}(f_g) \end{vmatrix}.$$

7.4.1 First steps

We first fix a basis $\{f_1, \dots, f_{g_\mathfrak{p}}\}$ of $M^2_{2,1}(\Gamma_0(\mathfrak{p})$ with integral u-series coefficients at ∞. As we remarked before, when computing the form $W(f_1, \dots, f_{g_\mathfrak{p}})$, one needs to compute D_n for $n < q^d$. Thus in all of the cases we will consider, we have that $f \equiv f' \pmod{\mathfrak{p}}$ implies that $D_n(f) \equiv D_n(f') \pmod{\mathfrak{p}}$ by Corollary 4.8.

Now suppose that $q \geq 3$. Under the correspondence exhibited in Chapter 6, we may produce a basis for the space $M^2_{q^d+1,1}(\mathrm{GL}_2(A))$ which we will denote by $\{F_1, \dots, F_{g_\mathfrak{p}}\}$, all of whose elements have integral u-series coefficients and such that $f_i \equiv F_i \pmod{\mathfrak{p}}$. Thanks to the considerations of the previous paragraph, we have

$$W(f_1, \dots, f_{g_\mathfrak{p}}) \equiv W(F_1, \dots, F_{g_\mathfrak{p}}) \pmod{\mathfrak{p}}.$$

Recalling the Serre operator $\partial^{(d)}_n$ from Chapter 4, we have that $D_n(f)$ and $\partial^{(k)}_n(f)$, for k the weight of f, differ by the sum

$$\sum_{i=1}^{n} (-1)^i \binom{k+n-1}{i} (D_{i-1}E)(D_{n-i}f).$$

We note that the quantity $(-1)^i\binom{k+n-1}{i}(D_{i-1}E)$ depends on k and n, but not on f. To ease notation, we write M_D for the matrix appearing in $W(F_1, \dots, F_{g_\mathfrak{p}})$, and M_∂ for the matrix appearing in the definition of $W_\partial(F_1, \dots, F_{g_\mathfrak{p}})$. Then we have that the $(n+1)$st column of M_∂ is equal to the $(n+1)$st column of M_D plus a linear combination of earlier columns of M_D. Since we are taking a determinant, we conclude that $W(F_1, \dots, F_{g_\mathfrak{p}}) = W_\partial(F_1, \dots, F_{g_\mathfrak{p}})$.

7.4.2 A special case

In order to proceed with a computation in a special case, we first restrict our attention to the case where $d = 3$. In that case $g_{\mathfrak{p}} = q$ and the canonical orders are $J = (1, \ldots, q-1)$.

Since the algebra of Drinfeld modular forms for $\mathrm{GL}_2(A)$ is generated by g and h, we have that

$$g^{n(q+1)} h^{q^2-q+1-n(q-1)}, \qquad 0 \leq n \leq q-1$$

is a basis for the space $M^2_{q^3+1,1}(\mathrm{GL}_2(A))$ with integral u-series coefficients, and there is a basis $\{f_1, f_2, ...f_{g_{\mathfrak{p}}}\}$ of $M^2_{2,1}(\Gamma_0(\mathfrak{p}))$ that maps to this basis under the trace map. Thus thanks to the work of the previous Section, we are interested in computing

$$W_\partial(h^{q^2-q+1}, \ldots, g^{q^2-1}h^q).$$

When $1 \leq n < p$ for p odd, we have that $\partial^n f = n!\partial_n^{(k)} f$, where as before the exponent of n on ∂ denotes the n-fold iteration. Therefore when $q = p$, the computation of $W_\partial(h^{p^2-p+1}, \ldots, g^{p^2-1}h^p)$ can be done easily using the fact that $\partial(g) = -h$ and $\partial(h) = 0$, and we get

$$W_\partial(h^{p^2-p+1}, \ldots, g^{p^2-1}h^p) = g^{\frac{p^2(p-1)}{2}} h^{\frac{p^2(p+1)}{2}}.$$

Tracing back through our steps, we have

$$\begin{aligned} aW(z) &= W(f_1, \ldots, f_p) \\ &\equiv W(h^{p^2-p+1}, \ldots, g^{p^2-1}h^p) \pmod{\mathfrak{p}} \\ &= W_\partial(h^{p^2-p+1}, \ldots, g^{p^2-1}h^p) \\ &= g^{\frac{p^2(p-1)}{2}} h^{\frac{p^2(p+1)}{2}}, \end{aligned}$$

for a rational and $\mathfrak{p}$-integral.

Denote by n_0 the index of the first non-zero u-series coefficient of $W(z)$. Then analytically, $W(z)$ vanishes to order $\frac{n_0}{p-1}$ at ∞. Thus the order of vanishing of $W(z)(dz)^{\frac{p(p+1)}{2}}$, is

$$\frac{n_0}{p-1} - \frac{p}{p-1}\left(\frac{p(p+1)}{2}\right).$$

Since this quantity must be non-negative, we have that $n_0 \geq \frac{p^2(p+1)}{2}$. At the same time, we have that the first non-zero u-series coefficient of $g^{\frac{p^2(p-1)}{2}}h^{\frac{p^2(p+1)}{2}}$ has index $\frac{p^2(p+1)}{2}$, so $n_0 = \frac{p^2(p+1)}{2}$. Since the leading coefficient of h is -1, the leading coefficient of $g^{\frac{p^2(p-1)}{2}}h^{\frac{p^2(p+1)}{2}}$ is $(-1)^{(p+1)/2}$. Thus we must have $a \equiv (-1)^{(p+1)/2} \pmod{\mathfrak{p}}$, from which we get our theorem:

Theorem 7.11. *If p is odd, $\pi \in \mathbb{F}_p[T]$ has degree 3, $\mathfrak{p}$ is the ideal generated by π, and the Wronskian on $X_0(\mathfrak{p})$ is denoted by $W(z)$, then we have*

$$W(z) \equiv (-1)^{(p+1)/2} g^{\frac{p^2(p-1)}{2}} h^{\frac{p^2(p+1)}{2}} \pmod{\mathfrak{p}}.$$

7.5 Obstacles to moving forward

Theorem 7.11 is an analogue of the theorem obtained by Rohrlich in the classical case (Theorem 1.1). As described in the Introduction, Ahlgren and Ono used this theorem to obtain the formula

$$\prod_{Q \in X_0(\ell)} (x - j(Q))^{\mathrm{wt}(Q)} \equiv \prod_{\substack{E/\overline{\mathbb{F}}_\ell \\ E \text{ supersingular}}} (x - j(E))^{g_\ell(g_\ell - 1)} \pmod{\ell}.$$

which precisely relates the Weierstrass points to the supersingular locus modulo ℓ. Our efforts to obtain a similar result met with some obstacles which we describe now.

We first describe the key idea in Ahlgren and Ono's work, omitting any discussion of the elliptic points since this would needlessly confuse the issue. Let f be a form of weight k and f' be a form of weight k', both for $\mathrm{SL}_2(\mathbb{Z})$, such that $f \equiv f' \pmod{\ell}$ and $k' < k$. Then for every P on $X_0(1)$ such that its reduction is supersingular modulo ℓ, we must have that f vanishes to order $\frac{k-k'}{\ell-1}$ at P. This is a consequence of the fact that the only relationship modulo ℓ in the algebra of modular forms for $\mathrm{SL}_2(\mathbb{Z})$ is $E_{\ell-1} \equiv 1 \pmod{\ell}$, where $E_{\ell-1}$ is the Eisenstein series of weight $\ell - 1$, and modulo ℓ, the divisor of $E_{\ell-1}$ is precisely the set of supersingular points, each with multiplicity 1. In short, if a form has lower filtration than weight, this must be explained by zeroes that belong to the supersingular locus modulo ℓ. The same facts, with appropriate modifications, are true in the Drinfeld setting, by Theorem 2.9.

To use this idea, it is crucial that both f and f' be forms on $\mathrm{SL}_2(\mathbb{Z})$. However, the form $W(z)$, which is defined by Ahlgren and Ono analogously to ours, is defined over $\Gamma_0(\ell)$. While the additive trace from $\Gamma_0(\ell)$ to $\mathrm{SL}_2(\mathbb{Z})$ (analogous to the additive trace map presented in Chapter 6) does give us a map from forms on $\Gamma_0(\ell)$ to forms on $\mathrm{SL}_2(\mathbb{Z})$, this map does not preserve any nice properties of the divisor. It is thus necessary to use the multiplicative trace, which for a form f of weight k and $|_k[\gamma]$ the classical slash operator, is given by

$$\mathrm{Tr}_m(f) = \prod_{\gamma \in \Gamma \backslash \mathrm{SL}_2(\mathbb{Z})} f|_k[\gamma].$$

This map has the property that for $\tau_0 \in \mathbb{H}$, the complex upper half-plane, we have

$$\mathrm{ord}_{\tau_0} \mathrm{Tr}_m(f) = \sum_{\substack{\tau \in \Gamma_0(\ell) \backslash \mathbb{H} \\ \tau \sim \tau_0}} \mathrm{ord}_\tau f,$$

where $\sim$ here denotes $\Gamma_0(\ell)$-equivalence.

In light of this, a key result obtained by Ahlgren and Ono is the following congruence:

$$\mathrm{Tr}_m(W(z)) \equiv W(z)^2 \pmod{\ell}. \tag{7.2}$$

Using Rohrlich's result, they can then show that $\mathrm{Tr}_m(W(z))$ is congruent modulo ℓ to a form of lower weight whose divisor modulo ℓ is essentially trivial (again we ignore elliptic points). The difference in weight is exactly $(g_\ell^2 - g_\ell)(\ell - 1)$, which explains the power of the supersingular polynomial appearing in the formula.

In the Drinfeld setting, it seems extremely hard to obtain a formula such as formula (7.2) at present. The first obstacle is the fact that product expansions of Drinfeld modular forms are not known in general. It is encouraging that Gekeler obtains in [17] a product expansion for the form Δ, and it is likely that such results could be generalized to overcome this obstacle.

The more serious obstacle is that the proof of (7.2) relies on the fact that the classical function $q(z) = e^{2\pi i z}$ is multiplicative, or more precisely on the fact that the new $\mathbb{Z}$-module structure given by the 1-dimensional lattice coincides with multiplication. We reproduce here a key step of the proof given by Ahlgren and Ono, where g is the genus of $X_0(\ell)$, $W(z) = q^{\frac{g(g+1)}{2}} \prod_{n=1}^{\infty}(1-q^n)^{c(n)}$ is the product expansion of $W(z)$ and $\zeta_\ell = e^{\frac{2\pi i}{\ell}}$. We then have:

$$\begin{aligned} q^{\frac{g(g+1)}{2}} \prod_{n=1}^{\infty}\prod_{j=0}^{\ell-1}(1-q^{\frac{n}{\ell}}\zeta_\ell^{nj})^{c(n)} &= q^{\frac{g(g+1)}{2}} \prod_{\ell\nmid n}(1-q^n)^{c(n)} \prod_{\ell\mid n}(1-q^{\frac{n}{\ell}})^{\ell c(n)} \\ &\equiv q^{\frac{g(g+1)}{2}} \prod_{n=1}^{\infty}(1-q^n)^{c(n)} \pmod{\ell}. \end{aligned}$$

In the Drinfeld setting, the A-module structure given by the Carlitz module does not coincide with multiplication, which makes the analogue of the congruence exhibited here difficult to prove. A completely different idea will be needed to address this issue.

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