§ R.1: Polynomials

Basic Definitions:

A polynomial is a term or finite sum of terms in which all variables have whole number exponents.

\[ 5x^4 + 2x^3 + 6x \]

Each piece of the expression are its terms (i.e. \(5x^4\)).

Here, \(5x^4\) has a coefficient of 5 and exponent of 4.

Terms having the same variable and exponent are like terms, if not they are called unlike terms.

\[ 5x^4 + 2x^4 + 3x^3 = 7x^4 + 3x^3 \]

\(6x^4 + 6p^4\) are unlike terms as well (different variables).

Order of Operations (PEMDAS)

When simplifying an algebraic expression follow this order:

Parentheses/ Powers: Start evaluating parenthesis and then powers (exponents) from left to right.

Multiplication/Division: again perform in order from left to right.

Addition/ Subtraction: in order from left to right.

\[ e.g.: (6(2+1)^2 + 3(e) - 22)^2 = (6(3)^2 + 6 - 22)^2 = (54 + 6 - 22)^2 = (38)^2 = 1444 \]
Properties of Real Numbers

\[ a, b, c \in \mathbb{R} \quad (\text{Real Numbers}) \]

1. Commutative: \[ a + b = b + a \quad \text{e.g.} \quad 2 + x = x + 2 \quad \text{and} \quad x \cdot 3 = 3x \]

2. Associative: \[ (a + b) + c = a + (b + c) \quad \text{e.g.} \quad (2 + 3) + 4 = 2 + (3 + 4) = 9 \quad \frac{2}{x} (x) = 7 (x \cdot x) = 7x^2 \]

3. Distributive: \[ a(b + c) = ab + ac \quad \text{e.g.} \quad 3(x + 4) = 3x + 12 \quad -(x - 2) = -x + 2 \]

Use these properties to simplify polynomials. Always combine like terms in your final simplified answer.

\[ \text{e.g.} \quad (8x^3 + 2x^2 + 3) - (3x^2 + 2) = \overrightarrow{8x^3} + \overrightarrow{2x^2} + \overrightarrow{3} - \overrightarrow{3x^2} - 2 = \overrightarrow{8x^3} - x^2 + 1 \]

Use the distributive property to multiply polynomials:

Note: \[ a^m \cdot a^n = a^{m+n} \quad \text{(i.e.)} \quad x^3 \cdot x^5 = x^{3+5} = x^8 \]

\[ \text{e.g.} \quad (3y)(2y - 4) = 6y^2 - 12y \]

For two binomials (2-term polynomials) use the F.O.I.L method:

\[ \text{e.g.} \quad (2x - 5)(x + 4) = 2x(x) + 2x(4) - 5(x) - 5(4) = 2x^2 + 3x - 20 \]

\[ \text{e.g.} \quad (2x - 5)(x + 4)(x + 2) = (2x^2 + 3x - 20)(x + 2) = 2x^3 + 4x^2 + 3x^2 + 20x - 20 \cdot 2x - 20 \cdot 2^2 = 2x^3 + 7x^2 + 14x - 40 \]

\[ \text{Remember: } (x + y)^n \neq x^n + y^n \text{ for } n \neq 1 \]
§ R.2 : Factoring

Using the distributive property to rewrite a polynomial as the product of other, smaller polynomials is called factoring.

E.g.: \( 18 = 9 \cdot 2 = 3 \cdot 3 \cdot 2 \); \( 2x^2 + 2x = 2x(x+2) \)

The largest factor \( # \) that can be factored out is called the **greatest common factor (gcf)**.

E.g.: \( 15m + 45 = 15(m + 3) \), here 15 is the gcf.

\( 2x^2 + x = x(2x+1) \)

When factoring a trinomial use F.O.I.L. backwards.

E.g.: \( y^2 + 8y + 15 \)

Since coefficient of \( y^2 \) (leading term) is 1 we need 2 numbers whose sum is 15 and product is 8.

<table>
<thead>
<tr>
<th>Product</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>15 \cdot 1 = 15</td>
<td>15 + 1 = 16</td>
</tr>
<tr>
<td>5 \cdot 3 = 15</td>
<td>5 + 3 = 8</td>
</tr>
</tbody>
</table>

\( y^2 + 8y + 15 = (y+5)(y+3) \)

When the coefficient of the leading term (highest degree) is not 1, you must also consider all possible factors of the 1st term and proceed by trial and error (we'll see a quicker way later).

**Special Factorizations**

1. \( x^2 - y^2 = (x+y)(x-y) \) **Difference of Squares**
2. \( x^2 + 2xy + y^2 = (x+y)^2 \) **Perfect Square**
3. \( x^3 - y^3 = (x-y)(x^2 + xy + y^2) \) **Difference of Cubes**
4. \( x^3 + y^3 = (x+y)(x^2 - xy + y^2) \) **Sum of Cubes**
e.g.1 Factoring Polynomials

(A) \(64r^2 - 4q^2 = (8r)^2 - (2q)^2 = (8r + 2q)(8r - 2q)\)

(B) \(x^2 + 36\) is a prime polynomial (cannot factor)

(C) \(x^2 + 12x + 36 = (x + 6)^2\)

(D) \(x^3 - 8 = x^3 - 2^3 = (x - 2)(x^2 + 2x + 4)\)

(E) \(p^4 - 1 = (p^2 + 1)(p^2 - 1) = (p^2 + 1)(p + 1)(p - 1)\)

When factoring, want each polynomial factor to be prime.
R.3 Rational Expressions

Rational Expressions are quotients of polynomials with non-zero denominators.

Properties: All mathematical expressions \( P, Q, R, S \) with \( Q \neq 0, S \neq 0 \)

1. \( \frac{P}{Q} = \frac{P S}{Q S} \) \( \quad \) Fundamental Property
2. \( \frac{P}{Q} + \frac{R}{Q} = \frac{P + R}{Q} \) \( \quad \) Addition
3. \( \frac{P}{Q} - \frac{R}{Q} = \frac{P - R}{Q} \) \( \quad \) Subtraction
4. \( \frac{P}{Q} \cdot \frac{R}{S} = \frac{P R}{Q S} \) \( \quad \) Multiplication
5. \( \frac{P}{Q} \div \frac{R}{S} = \frac{P \cdot S}{Q \cdot R} \) \( (R \neq 0) \) \( \quad \) Division

6. \( \frac{a^m}{a^n} = a^{m-n} \) \( \quad \) (i.e. \( \frac{x^4}{3x} = \frac{1}{3} \cdot \frac{x^4}{x} = \frac{1}{3} \cdot x^{4-1} = \frac{1}{3} \cdot x^3 \))

E.g. \( \frac{m^2 + 5m + 6}{m + 3} \cdot \frac{m}{m^2 + 3m + 2} = \frac{(m+2)(m+3)}{(m+1)(m+2)} \cdot \frac{m}{m+2} = \frac{m}{m+1} \)

E.g. \( \frac{9p - 36}{12} \div \frac{5(p-4)}{18} \) \( \quad \) (Invert and Multiply !)

\( \Rightarrow \frac{9p - 36}{12} \cdot \frac{18}{5(p-4)} = \frac{9}{12} \cdot \frac{18}{5(p-4)} = \frac{9 \cdot 18}{12 \cdot 5} \)

\( = \frac{9 \cdot 6 \cdot 3}{6 \cdot 2 \cdot 5} = \frac{27}{10} \)
C.3.1 \[ \frac{7}{P} + \frac{9}{2P} + \frac{1}{3P} \] To combine each term we need a least common denominator, hence \( \text{lcm} = 6P \)

\[ \Rightarrow \frac{7 \cdot 6}{P \cdot 6} + \frac{9 \cdot 3}{2P \cdot 3} + \frac{1 \cdot 2}{3P \cdot 2} = \frac{42}{6P} + \frac{27}{6P} + \frac{2}{6P} = \frac{71}{4} \]

C.3.3 \[ \frac{x+1}{x^2+5x+6} - \frac{5x-1}{x^2-x-12} = \frac{x+1}{(x+3)(x+2)} - \frac{5x-1}{(x+3)(x-4)} \]

\[ = \frac{x+1}{(x+3)(x+2)} \cdot \frac{(x-4)}{(x-4)} - \frac{5x-1}{(x+3)(x-4)} \cdot \frac{x+2}{x+2} \]

\[ = \frac{(x+1)(x-4) - (5x-1)(x+2)}{(x+2)(x+3)(x-4)} = \frac{x^2-3x-4 - (5x^2+9x-2)}{(x+2)(x+3)(x-4)} \]

\[ = \frac{-4x^2-12x-12}{(x+2)(x+3)(x-4)} = \frac{-2(2x^2+6x+1)}{(x+2)(x+3)(x-4)} \]
§R.4: Equations

**Linear Equations**: eqn's of the form \( ax + b = 0 \) with \( a \neq 0 \)

**Properties of Equality**: \( \forall a, b, c \in \mathbb{R} \)

1. \( a = b \implies a + c = b + c \) (Addition)
2. \( a = b \implies ac = bc \) (Multiplication)

*Example:*

\[
2x - 5 + 8 = 3x + 2(2 - 3x)
\]

\[
\iff 2x + 3 = 3x + 4 - 6x \iff 2x + 3 = 4 - 3x
\]

\[
\iff 2x + 3x = 4 - 3 \iff 5x = 1 \iff x = \frac{1}{5}
\]

**Quadratic Eqn**: Equations whose leading term has power 2 and are of the form \( ax^2 + bx + c = 0 \) with \( a \neq 0 \)

This is called the standard form

**Zero Factor Property**: \( \forall a, b \in \mathbb{R}, \ ab = 0 \implies a = 0, b = 0 \) or both

Either factor the equation and use the zero factor property or use the quadratic formula

**Quadratic Formula**

The solutions of the quadratic equation \( ax^2 + bx + c = 0, a \neq 0 \) are given by:

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]
C. 1. Solve $x^2 - 4x = 5$

1. Rewrite in standard form: $x^2 - 4x - 5 = 0$

**Factor Approach:** $x^2 - 4x - 5 = (x + 1)(x - 5) = 0 \Rightarrow x = 5$ or $x = -1$  
(Zero-factors property)

**Quadratic Formula:** Here $a = 1, b = -4, c = -5$

$$\Rightarrow x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(-5)}}{2} = \frac{4 \pm \sqrt{16 + 20}}{2} = \frac{4 \pm \sqrt{36}}{2}$$

$$= \frac{4 \pm 6}{2} = 2 \pm 3 \Rightarrow \boxed{x_1 = 5, x_2 = -1}$$

Perfect to have radicals in your answer, however if there is a negative number under the radicand (i.e. $3 + 5\sqrt{}$)

then this is an example of a complex #

Sometimes you can solve quadratics directly:

$$e.g.\frac{2}{x^2} - 12 = 0 \Rightarrow \frac{2}{x^2} = 12 \Rightarrow 12x^2 = 2$$

$$\Rightarrow \frac{2}{3} = \frac{1}{4} \Rightarrow x = -\frac{1}{2} \pm \frac{1}{2}$$

It is possible to obtain a solution to a rational equation that makes the denominator = 0, which is NOT allowed, called Extraneous Solution

- **Solution**

$$e.g.\frac{2}{x} + \frac{1}{x} = \frac{6}{x(x-3)} \Rightarrow \frac{2x + x - 3}{x(x-3)} = \frac{6}{x(x-3)} \Rightarrow 2x + x - 3 = 6 \Rightarrow 3x = 9 \Rightarrow x = 3$$

However, plugging $x = 3$ into the original equation makes 2 denominator = 0, so we say No Solution or $\emptyset$ for this equation.
§R.5: Inequalities

Inequality Symbols: \( \forall a,b,c \in \mathbb{R} \)

1. if \( a < b \) then \( a + c < b + c \)
2. if \( a < b \) and \( c > 0 \), then \( ac < bc \)
3. if \( a < b \) and \( c < 0 \), then \( ac > bc \) (Sign flip! e.g. \( 2 < 3 \) but \( -2 \geq -3 \))

E.g.: Linear Inequality: Solve
\[
4 - 3y \leq 7 + 2y \iff -5y \leq 3 \iff y \geq -\frac{3}{5}
\]

You can graph the solution on a number line:

\[ -1 \quad \frac{3}{5} \quad 0 \quad 1 \]

E.g.: \( \frac{-7}{3} < m < 5 \)

Interval Notation

\( a \leq x \leq b \equiv x \in [a, b] \equiv "x \text{ is an element of the closed interval } [a, b]" \)

\( a < x < b \equiv x \in (a, b) \equiv "x \text{ is in the open interval on } (a, b)" \)

Intervals can also be half-open and half closed

E.g.: Quadratic Inequalities: Solve \( x^2 - x < 12 \)

Rewrite the inequality as: \( x^2 - x - 12 < 0 \)

Now solve \( x^2 - x - 12 = 0 \)

Factor \((x-4)(x+3) = 0 \iff x = 4 \text{ or } x = -3\)

Now at each of these points, there can be a sign change, so we must consider three intervals

\( (-\infty, -3), [-3, 4], (4, \infty) \)

\[ \begin{array}{cccc}
\text{A} & 0 & \text{C} \\
-3 & 4 & -3
\end{array} \]
Now choose points from each of these intervals and plug into \( x^3 - x - 12 \).

\( A = (-\infty, -3) \) choose \(-4\) \( (-4)^2 - (-4) - 12 = 8 > 0 \) \( \times \) Not Satisfied

\( B = [-3, 4] \) choose 0 \( (0)^2 - (0) - 12 = -12 < 0 \) \( \checkmark \) Satisfied

\( C = (4, \infty) \) choose 5 \( (5)^2 - (5) - 12 = 8 > 0 \) \( \times \) Not Satisfied

So the solution is given by the interval \((-3, 4)\).

\[ \boxed{\text{E.g. Rational Inequality: Solve } \frac{2x-3}{x} \geq 1} \]

First we want to see when this expression is equal to the right hand side (1).

Since this will tell us the critical point:

\[ \frac{2x-3}{x} = 1 \iff 2x-3 = x \iff x = 3 \]

Now there can also be a sign change when the denominator changes from negative to positive. Thus we must consider when the denominator equals 0 as another critical point \( x = 0 \).

That gives us 3 intervals to check: \((-\infty, 0), [0, 3], [3, \infty)\).

\[ (-\infty, 0) \] choose \(-1\) \( \iff \frac{2(-1)-3}{-1} = 5 \geq 1 \) \( \checkmark \)

Check endpoint: \( 0 \) \( \iff \frac{2(0)-3}{0} = -\infty \) \( \times \)

\[ [0, 3) \] choose 1 \( \iff \frac{2(1)-3}{1} = -1 \neq 1 \) \( \times \)

Check endpoint: \( 3 \) \( \iff \frac{2(3)-3}{3} = 1 \geq 1 \) \( \checkmark \)

\[ [3, \infty) \] choose 4 \( \iff \frac{2(4)-3}{4} = \frac{5}{4} \geq 1 \) \( \checkmark \)

The solution is given by \( \boxed{(-\infty, 0) \cup [3, \infty)} \).
§R.6 Exponents

Definition: if \( n \) is a natural \( \mathbb{N} = \{1, 2, \ldots \} \), then
\[
\begin{align*}
a^n &= a \cdot a \cdot a \cdots a \quad \text{n times} \quad ; \quad a^0 &= 1 \quad ; \quad a^{-n} &= \frac{1}{a^n}
\end{align*}
\]

Properties: For \( m, n \in \mathbb{Z} \), \( a, b \in \mathbb{R} \)

1. \( a^m \cdot a^n = a^{m+n} \)
2. \( \frac{a^m}{a^n} = a^{m-n} \)
3. \( (a^m)^n = a^{mn} \)
4. \( (ab)^m = a^m \cdot b^m \)
5. \( \left( \frac{a}{b} \right)^m = \frac{a^m}{b^m} \)
6. \( (-a)^n = \begin{cases} a^n & \text{if } n \text{ is even integer } \mathbb{Z}^+ \\ -a^n & \text{if } n \text{ is an odd integer } \mathbb{Z} \end{cases} \)

C.3.1 Simplifying Exponential expressions:

1. \( 3 \cdot 2^2 \cdot 4^2 = 3 \cdot 4 \cdot 16 = 192 \)
2. \( (2m^3)^4 = 2^4 \cdot m^{12} = 16m^{12} \)
3. \( \left( \frac{x^2}{y^3} \right)^6 = \frac{x^{12}}{y^{18}} \)
4. \( \frac{a^3 b^5}{a^2 b^3} = \frac{b^{12}}{a^{12}} \)
5. \( \frac{x^2 - y^2}{x^2 - y^2} = \frac{x^2 - y^2}{x^2 - y^2} = \frac{y^2}{x^2} = \frac{x^2}{y^2} = \left( \frac{x}{y} \right)^2 \)

Exponents can also be rational numbers, or roots.
The most familiar is: \( \sqrt[n]{x} = x^{\frac{1}{n}} \)

which can generalize to:

\[
\begin{align*}
\alpha^{m/n} &= (\alpha^{1/n})^m, \quad \forall \alpha \in \mathbb{R}, \ m, n \in \mathbb{Z}
\end{align*}
\]

e.g.: \( 2^{\frac{2}{3}} = (2^{\frac{1}{3}})^2 = (3^{\frac{1}{3}})^2 = 3^2 = 9 \)
§R.7: Radicals

Instead of exponents we can also use radical notation:

Radicals: If $n$ is an even natural # and $a > 0$ or $n$ is an odd natural # then

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$

Properties: \(n, m, a, b \in \mathbb{N}\)

1. \(\left(\sqrt[n]{a}\right)^n = a\)

2. \(\sqrt[n]{a^n} = \begin{cases} a & \text{if } n \text{ is even} \\ |a| & \text{if } n \text{ is odd} \end{cases}\)

3. \(\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}\)

4. \(\sqrt[n]{a^m} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} (b \neq 0)\)

5. \(\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}\)

Note property 2 due to the fact:

\((-a)^n = -a^n \text{ if } n \text{ is odd and } (-a)^n = a^n \text{ if } n \text{ is even}\)

Defn: The absolute value of $x$ is defined as

\(|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}\)

Note: $\sqrt[n]{a^n + b^n} \neq \sqrt[n]{a^n} + \sqrt[n]{b^n}$

Sometimes it's useful to remove the radical from either the numerator or denominator of an expression, we call this rationalization.

Eg. \(\frac{4}{\sqrt{3}}\) To rationalize the denominator, simply multiply by \(\frac{\sqrt{3}}{\sqrt{3}}\).

\(\frac{4}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{4 \cdot \sqrt{3}}{(\sqrt{3})^2}\)

\((a)\) \(1 - \sqrt{2} = \frac{1}{1 - \sqrt{2}} \cdot \frac{(1 + \sqrt{2})}{(1 + \sqrt{2})} = \frac{1 + \sqrt{2}}{1 + (\sqrt{2}^2 - 2)} = \frac{1 + \sqrt{2}}{1} = 1 - \sqrt{2}\)

\((b)\) \(\frac{2}{\sqrt{x}} = \frac{2}{\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}} = \frac{2\sqrt{x^2}}{\sqrt{x^2}} = \frac{2x}{x} = 2\)