1 Cobb-Douglas Functions

Cobb-Douglas functions are used for both *production functions*

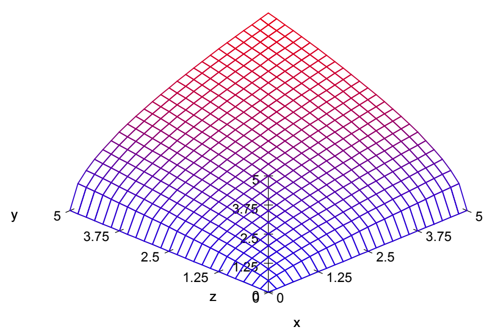
$$Q = K^{\beta} L^{(1-\beta)}$$

where Q is output, and K is capital and L is labor. The same functional form is also used for the utility function; we often write:

$$U = X^{\beta} Y^{(1-\beta)}$$

where X and Y are two different *goods.* These two expressions are mathematically equivalent.

The Cobb-Douglas function is *three dimensional* with utility or output measured along the vertical axis.



We rarely work with 3-D graphs such as this however; instead we slice it into various views or *projections.* The projection onto the XY axis is called an *indifference curve.* The projection is made by slicing off the top of the three-dimensional utility surface and shinning a light from above. The shadow that the edge projects on the floor is the indifference curve. These projections are useful for graphical purposes only; when working with them, we must remember that we are still using the full Cobb-Douglas equation.

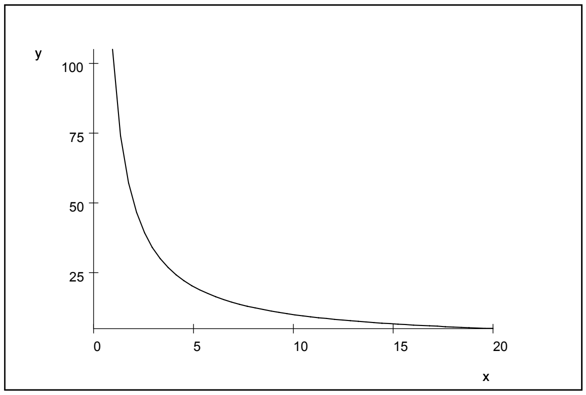
To get the XY projection in the utility function, we simply take a *given level of* $U = U_0$ and solve for Y as a function of X

$$U_0 = X^{\beta} Y^{(1-\beta)}$$

Solution is:

$$Y = U_0^{\frac{1}{1-\beta}} X^{\frac{-\beta}{1-\beta}}$$

but since $U_0^{\frac{1}{1-\beta}}$ is a constant along the projection, this is simply an *hyperbola.* It has a graph that looks like:

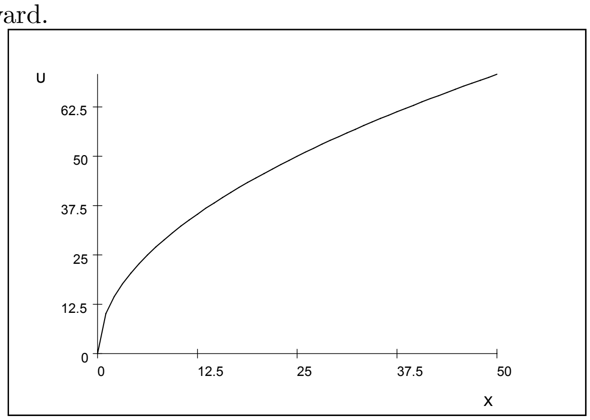


which is an *indifference curve*, or combinations of X and Y such that the level of utility is a constant, U_0. For $U_1 > U_0$ the curve would shift out from the origin, but along the curve, total utility is constant. Indifference curves are similar to topographical maps; all along an indifference curve, we are at the same altitude above the floor.

The projection in the X, U plane can be also be plotted and studied as in the following figure. To get this graph start with a given level of Y and plot

$$U = X^{\beta}\bar{Y}^{(1-\beta)}$$

where the bar over Y indicates that Y is constant. Increasing Y would shift the curve upward.



and a similar diagram can be drawn for the projection into the Y, U plane.

1.1 Slopes

For the utility function, the *slope* of this curve in the X, U plane is just the *marginal utility of* X, holding Y constant. For the production function, the slope is the *marginal product* of one of the two factors, holding the other constant. Using calculus, the slope is simply the partial derivative of the Cobb-

Douglas function with respect to X holding Y constant, or vice-versa.

$$\frac{\partial U}{\partial X}_{|Y=\bar{Y}} = \text{ marginal utility of } X$$

One of the reasons the Cobb-Douglas is so popular is that its derivatives are so *simple*. The function itself is not, but taking the partial of U with respect to X gives:

$$\frac{\partial U}{\partial X} = \beta X^{\beta-1} Y^{1-\beta} = \frac{\beta U}{X}$$

With respect to Y

$$\frac{\partial U}{\partial Y} = (1-\beta) X^{\beta} Y^{1-\beta-1} = \frac{(1-\beta) U}{Y}$$

1.1.1 The slope of the horizontal projection

The horizontal projection of the utility function is an indifference curve. For a production function, it is an isoquant. The slope of either of these two curves in just the rise over the run as it would be in any 2-D space.

For the square root version of the Cobb-Douglas, $\beta = 0.5$, the slope is very easy to calculate. Consider an indifference curve. First pick a point on the curve, call it X_1, Y_1. On the same indifference curve a second point might be $X_2 Y_2$ and think of it has lower and to the right of the first point. The *run* is just $X_2 - X_1$ while the rise is $Y_2 - Y_1$. But since Y_2 is less than Y_1 the *rise* is negative.

Since both points are on the indifference curve, the utility must be the same; we have:

$$U(X_1, Y_1) = U(X_2, Y_2)$$

Subtracting, we can write:

$$U_1^2 - U_2^2 = 0$$

where $U_i = \sqrt{X_i, Y_i}$. Finally, rewrite this last equation as:

$$X_1 Y_1 - X_2 Y_2 = 0$$

To get the slope of the indifference curve, we can just *add and subtract* $X_1 Y_2$ to the last equation and write the result as:

$$X_1 Y_1 - X_2 Y_2 + X_1 Y_2 - X_1 Y_2 = 0$$

This is a *key substitution,* one that only works for the square root version of the Cobb-Douglas. Reorganizing this last expression

$$X_1 (Y_1 - Y_2) + (X_1 - X_2) Y_2 = 0$$

The elements a slope, the rise $(Y_1 - Y_2)$ over run $(X_1 - X_2)$ are starting to take shape. Divide both sides by $(X_1 - X_2)$

$$X_1 \frac{(Y_1 - Y_2)}{(X_1 - X_2)} + Y_2 = 0$$

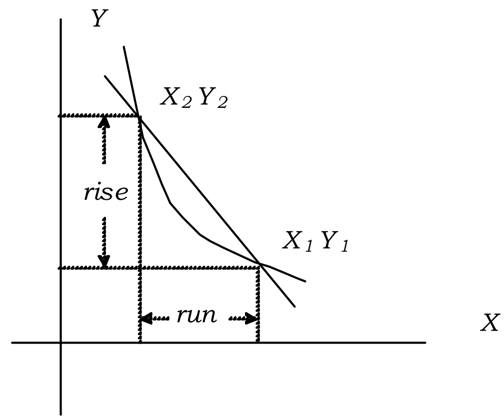


Figure 1:

and call the slope, σ

$$\sigma X_1 + Y_2 = 0$$

and solve for $\sigma = -\frac{Y_2}{X_1}$

This is the slope of the straight line in the graph above. To get the instantaneous slope at the point X_1Y_1, just move the two points closer and closer together, that is move X_2Y_2 down toward X_1Y_1. In the limit, that is as the distance between the points goes to zero, we have:

$$\sigma = -\frac{Y}{X}$$

The slope is known as the *marginal rate of substitution.*

In the case of the *isoquant*, the argument is identical; instead of X and Y we have K and L. But this gives:

$$\sigma = -\frac{K}{L}$$

so long as L is on the horizontal axis, taking the place of X. For the isoquant, the slope is known as the *marginal rate of technical substitution.*

Another way to get slopes of horizontal projections is to use multivariate calculus. It gets the same result, but requires that you understand a *total differential* of a function of two variables. In the case of utility, the total differential of U is:

$$dU = \frac{\partial U}{\partial X} dX + \frac{\partial U}{\partial Y} dY$$

where $\frac{\partial U}{\partial X}$ is a *partial derivative,* that is the derivative of U with respect to X holding Y constant, said "the partial of U with respect to Y." [1] As we saw above, along an indifference curve or an isoquant, the change in U, $dU = 0$. We then have:

$$0 = \frac{\partial U}{\partial X} dX + \frac{\partial U}{\partial Y} dY$$

from which the slope, dY/dX of the indifference curve can be calculated.

$$\frac{dY}{dX} = -\frac{\partial U/\partial X}{\partial U/\partial Y}$$

but since the partial derivative of the Cobb-Douglas function is just

$$\frac{\partial U}{\partial X} = \beta X^{\beta-1} Y^{1-\beta} = \frac{\beta U}{X}$$

and with respect to Y

$$\frac{\partial U}{\partial Y} = (1-\beta) X^{\beta} Y^{1-\beta-1} = \frac{(1-\beta)U}{Y}$$

we have substituting into the definition of the slope:

$$\frac{dY}{dX} = -\frac{\beta UY}{(1-\beta)UX} = -\frac{\beta Y}{(1-\beta)X}$$

This is the general expression for the slope of the two-dimensional projection of the the Cobb-Douglas equation into the XY plane. In the special case of the square root function ($\beta = 0.5$) we have:

$$\frac{dY}{dX} == -\frac{Y}{X}$$

which is the same as the expression above.

1.2 Using the Cobb-Douglas for Utility or Profit Maximization

Now whether we are talking about maximizing utility or minimizing cost, setting the slope of the 2-D projection is set equal to the slope of the constraint is gives one equation for the solution to the maximization problem.

For example, in the maximization of utility problem the slope of the budget constraint is just the (negative of the) opportunity cost of X in terms of the good

[1] Similarly, the derivative of U with respect to Y holding X constant is $\frac{\partial U}{\partial Y}$.

Y, in other words, p_X/p_Y, the price of X divided by the price of Y. Maximizing utility is requires that these two slopes are the same:

$$-\frac{Y}{X} = -p_X/p_Y$$

This is the *tangency condition* which must be solved simultaneously with the *budget constraint* in order to find a maximum:

$$B = p_X X + p_Y Y$$

where B is the budget. Substituting the tangency condition into the budget constraint for Y, we have:

$$B = p_X X + p_Y(\frac{p_X}{p_Y} X)$$

Simplifying:

$$\begin{aligned} B &= 2p_X X \\ X &= B/2p_X \end{aligned}$$

and finally, substituting X into the budget constraint

$$B = p_X(B/2p_X) + p_Y Y$$

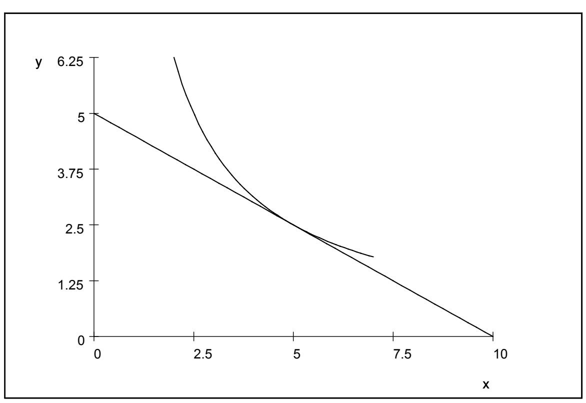
or simplifying

$$Y = B/2p_Y$$

So the solution involves setting dividing the budget into to two shares depending on their prices.

Example 1 ***Solution 2*** ***Problem 3*** ***Problem 4*** *Let* $p_X = 1$ *and* $p_Y = 2$ *and* $B = 10$. *Solve the consumer's maximization problem.*

Solution 5 $X = B/2p_X = 10/[2(1)] = 5$; *and* $Y = B/2p_Y = 10/[2(2)] = 2.5$. *Check to see that the budget is exhausted. Total utility is* $U = \sqrt{12.5}$ *The graphical solution is presented below:*



Example 6 ***Solution 7*** ***Problem 8*** ***Problem 9*** *Let* $w = 1$ *and* $r = 2$ *and* $C = 6$. *Solve the producer's maximization problem.*

Solution 10 *To maximize profits relative to a budget constraint we set slope* σ *of the cost constraint* $C = wL + rK$, *which is*

$$\sigma = -w/r$$

equal to the slope of the isoquant

$$dK/dL = -K/L$$

and solve simultaneously with the cost constraint itself. The structure of this problem is identical to that of consumer's problem. The are mathematically equivalent. Solving in the same was as in the previous example, $L = C/2w = 6/[2(1)] = 3$; *and* $K = C/2r = 6/[2(2)] = 1.5$. *Check to see that the cost constraint is exhausted. Output is* $Q = \sqrt{KL} = \sqrt{4.5}$