

Check your understanding: Solow model¹

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Solow model

The characteristics of the Solow model are²

- Solow has two kinds of variables, *state* variables that move slowly, between periods.
- *Jump* variables that move quickly, within one period.
- There are two versions
 1. Extensive: K is the state variable and labor growth $n = 0$
 2. Intensive: $k = K/L$ is the state variable and labor growth $n > 0$
- Production is demand driven and usually modeled with a Cobb-Douglas.
- Hats are important in using and understanding the model. Review the CYU for hats.
- The model has both a **transient** and **steady-state** solution in which the state variable stops growing.
- The intensive model in mathematical form
 1. GDP, Y is the sum of consumption and investment

$$Y = C + I$$

there is no government or foreign trade

2. The production function is Cobb Douglas³

$$Y = \mathcal{A}K^\beta L^{(1-\beta)} \quad (1)$$

with \mathcal{A} an arbitrary constant to model exogenous technical, K the capital stock, L is labor and β is the share of output going to capital if factors are paid their marginal products.

3. Investment is a constant fraction, s , of GDP

$$I = sY \quad (2)$$

4. The capital-labor ratio, k , is given by

$$k = K/L \quad (3)$$

and per worker income is given by

$$y = Y/L$$

¹ Thanks to Farzad Ashouri

² This is a simplified introduction. There is no technical change and no human capital in the model here, even though most references will develop the model for these and many other applications.

³ This function exhibits constant returns to scale, diminishing returns to labor and diminishing returns to capital. It is the last assumption that is responsible for the existence of a steady-state.

5. Capital depreciates at rate δ . This is a hat, but is negative and just says that capital shrinks from wear and tear as well as economic obsolescence.
6. The change in capital is then given by investment less depreciation

$$\Delta K = I - \delta K \quad (4)$$

This last equation is called the equation of motion of the Solow model and shows how the state variable changes over time.⁴

The model is made up of 5 equations. The variable list is then $V(Y, K, I, k, y)$ while the parameter list is $P(\mathcal{A}, \beta, L, s, \delta)$

⁴ It is in the same form as the “pond” equation with the change in the size of the pond equal to the inflow from the creek (investment) minus the outflow from evaporation, which is proportional to the size of the pond.

Deriving the Solow equation

Equation 4 is in *extensive form*. It takes some effort to develop the Solow equation for the intensive form case. Start by taking the hats of equation 3.

$$\hat{k} = \hat{K} - \hat{L} \quad (5)$$

Employment growth \hat{L} is usually taken proportional to the population growth. This assume that the labor force participation rate, l_f is constant and the rate of unemployment, u , is constant. Thus

$$L = (1 - u)l_f N$$

where N is the population. Taking hats of this equation, it is evident that if the unemployment rate is constant (or approximately so) and the labor force participation rate is constant (or approximately so) the rate of growth of employment is approximately equal to the rate of growth of the population, which is taken as given and equal to n . Equation 5 can then be written

$$\hat{k} = \Delta K/K - n$$

where \hat{K} is written as what it is, the percent change of the capital stock. Now substitute the equation of motion into this last equation to get

$$\hat{k} = (I - \delta K)/K - n$$

which simplifies to

$$\hat{k} = \frac{I}{K} - (\delta + n)$$

Next substitute equation 2 to get

$$\hat{k} = \frac{sY}{K} - (\delta + n)$$

and then divide the numerator and denominator of the first term on the right by L

$$\hat{k} = \frac{sY/L}{K/L} - (\delta + n)$$

Noting that $\hat{k} = \Delta k/k$ and that $y = Y/L$ from equation 4

$$\Delta k/k = \frac{sy}{k} - (\delta + n)$$

This is the *Solow equation* and is usually written as

$$\Delta k = sy - (\delta + n)k$$

The term on the left is called the *capital deepening term* while the first term on the right is available savings out of income per worker. The last term on the right is called the *capital widening term*. Rearranging the Solow equation gives a nice interpretation

$$sy = \Delta k + (\delta + n)k$$

Savings out of income per worker is responsible for both capital deepening (more capital per worker) and capital widening, replacing worn out capital and equipping new members of the labor force with the same capital that all existing workers have.

Transient and steady state solutions

In the steady state, the capital deepening term is zero. No more capital per worker is accumulated. All savings goes for capital widening. The steady state, $\Delta k = 0$ and

$$sy = (\delta + n)k$$

To get an expression for y from the Cobb-Douglas equation 1, simply divide by L ⁵

$$Y/L = \frac{AK^\beta L^{(1-\beta)}}{L}$$

which by the rules of exponents simplifies to

$$y = Ak^\beta$$

Substituting this into equation

$$sAk^{\beta-1} = (\delta + n)$$

or rearranging terms

$$\frac{sA}{\delta + n} = k^{1-\beta}$$

⁵ See the CYU on exponents.

The steady state k_{ss} can then be written as

$$k_{ss} = \left(\frac{sA}{\delta + n} \right)^{1/(1-\beta)}$$

and the steady state income per worker as

$$y_{ss} = \left(\frac{sA}{\delta + n} \right)^{\beta/(1-\beta)}$$

⁶ This is one of the most remarkable equations in all of economics. It says that economic output per worker can be determined by *five simple parameters*.⁷ The first is the savings rate, s . If one country saves more than another, it will be *richer* in the steady-state. This is in stark opposition to the Keynesian model that holds that more savings reduces demand and output and is the so-called “paradox of thrift.”⁸ The second parameter is the depreciation rate; faster wear and tear reduces income per worker as does faster economic obsolescence. This is undeniable and just part of the nature of production using capital equipment. The rate of growth of the population, n , is the next parameter. It has been taken as given here, but nothing suggests that it must remain constant. The *demographic transition* is one of the biggest changes that all fast growing countries must experience. Fertility after the demographic is lower than before, meaning that women have fewer children. They can therefore invest more in the human capital of each of their children. This increases the supply of skilled labor in the economy, lowers the skilled labor premium over unskilled labor. The latter improves the competitiveness of the economy and improves both the distribution of income as well as poverty indices. Countries that have non-inclusive institutions, such as Muslim theocracies, for example, will remain poor according to the Solow model. Since they discriminate against women, discouraging them from entering the labor force it becomes more difficult to invest in their children’s education and all of the advantages of the demographic transition are delayed.

The fourth parameter is the β the share of capital in total output. This is set by the relationship between the rate of increase in nominal wages, w , versus prices, p . If wage rise faster than the price level, the real wage will rise and the share of labor will increase. This will cause β to fall and for the economy to have a lower level of steady-state capital stock, and thus it will be poorer.

Finally, Solow got the Nobel prize for seeing more clearly than anyone else that diminishing returns to capital would, if given enough time, would strip capital of its ability to keep increasing output. The key is to understand that if capital is growing at 3 percent but labor is growing a 2 percent, diminishing returns is cause the rate of increase of output to fall. If these growth rates persist, capital will

⁶ Note that if $\beta = 0.5$, $\beta/(1-\beta) = 1$

⁷ Output per worker is more commonly known as *productivity*.

⁸ This is less of a paradox and more of just the conflict between consumption and savings of which every human—and many non-human species—on planet Earth is aware. Savings is painful in the short-run, but highly beneficial in the long run.

eventually become powerless to increase output in the same way that adding more and more shovels for workers digging holes will eventually cause output to stop growing. In this case, workers could also be increasing in some rate, but if the shovels are increasing faster, the system will reach a steady state.

If output per worker eventually stops because of diminishing returns, there is only one parameter left to saving the economy from long-term stagnation (zero or slow, or “vegetative” growth) is technological change captured in A . In *endogenous growth models* the agents in the economy have control some over A by devoting more resources to increase its growth rate. For exogenous growth models, such as the Solow model, technological change just happens. After k stops growing the only parameter that could continue to increase output per capita is A .

During the traverse, things are completely different. With capital deepening, output per worker is always growing. The closer the economy gets to the steady-state the more slowly output per worker grows. The same 5 parameters are responsible. To see this note that

$$\hat{y} = \beta \hat{k}$$

so that if $\beta = 0.5$ and capital per worker is growing at 3 percent, income per worker, productivity, is growing at 1.5 percent.⁹ To find out how fast k is growing, use the equation

$$\hat{k} = \Delta k/k = \frac{sy}{k} - (\delta + n)$$

from which it is obvious the an increase in s will speed up \hat{k} .

Policy

Policymakers can do some things to help growth but it cannot control all the five parameters equally or easily. It is difficult to change the savings rate of an economy. What should the central bank do? Raise the interest rate? Not really since that will discourage investment in the short run. In China the savings rate is high, but that is because it has no social safety nets to help people in retirement. This help raise the savings rate, but it has unintended consequences for human welfare if some people do not end up saving enough.

Policymakers might have most control over, n , the rate of population growth. China introduced the one-child policy and this certainly helped raise their rate of growth. This set up a virtuous circle in that the faster growth caused the Chinese economy to approach the demographic transition more rapidly and this *itself* caused the population growth rate to fall.

⁹ Again, see the CYU on hats.

The government probably has some control over β the share of capital. Reducing the power of labor unions helps since this restricts real wage growth. This is counterproductive in the Keynesian perspective since workers spend more of their income than those who derive their income from profits. In the Solow model, lower real wages speeds growth because β is higher. Workers tend to be the majority in most countries however, and unless that they see that they will be better off with lower wages now and more investment and growth, they will tend to vote for politicians who support "living wages" and other populist measures that make all workers worse off in the future.

Finally, raising the rate of technological progress causes faster growth in the traverse *as well as in the steady state*. No other parameter can do this. The reason is that the other parameters *cannot grow continuously*. Population growth rates around the world are steady or falling slightly. Savings rate also seem to be stable or falling slightly. The only real ability to raise growth would seem to be from stimulating technological through public funding for research and development. Agencies such as the National Institute of Health, National Science Foundation, The Defense Advanced Research Foundation and many other public (and private) organizations devoted to the production of public goods simulate \mathcal{A} . Even here, however, there are unintended consequences. Policymakers have almost no direct control over the rate of depreciation. Even stimulating technological change might be counterproductive if the technological change causes the rate of depreciation to rise because of faster obsolescence. This effect is probably quite small in any event.

Summary

1. In the steady state, there is no growth without technological change.
2. Getting to the steady state may take a long, long time, as much as 150 years.
3. One in the steady-state, raising the rate of savings will have no effect on growth since all growth depends on technological change.
4. The rate of growth of income depends on the rate of growth of the \mathcal{A} times the share of capital β .
5. Increasing the savings rate does increase growth in the traverse between steady states.

Graphing the Solow model

First plot capital widening for the intensive Solow model in figure 1. This figure has a slope of $(n + \delta)$ which for any given level of k shows the amount of savings necessary for depreciation and to equip new members of the labor force with capital.

Next plot capital deepening for the intensive Solow model in figure 2. This is the Cobb-Douglas production function written in intensive form *and then multiplied by the savings rate*.

The Solow model combines these two figures 3. The intersection of the two gives the Solow steady state. This long-run equilibrium is *stable* in the sense that to the left of the steady state, the available savings is greater than the needs of capital widening and so capital per worker is increasing. Capital deepening is taking place. To the right of the steady state, the need for capital widening is *greater* than the available savings and so capital deepening is *negative*. The economy is shedding capital.

These graphs can be used to picture the results above.

Convergence

Assuming that all countries have the same parameters \mathcal{A} , δ , n , s and β **capital per unit of labor** k should all converge to the same amount. Is that happening in the world? Prior to the rise of the Chinese economy, one would have to conclude that no, it is not. Poor countries seemed to remain poor and rich countries increased their well being. In the last decades, however, economists have started to think that convergence *cannot be ruled out*.

Here is the argument. While it is possible that different cultures might have different population growth rates, the demographic transition observe in *all* rich countries suggests that n might well converge across the globe. Technology is already shared so that there is no reason to think that some countries would always lag behind in adopting the best techniques. Shares of capital and labor depend on the *dominance* of the market mechanism. This also seems to be much deeply rooted today than in the past. To the extent that countries adopt populist policies and they work and therefore remain in force, β 's could converge as well. Since depreciation is largely a function of the shared technology, there is no real reason why these would differ in the long run.

Finally, savings rates could differ; but here there is an observable convergence since most countries adopt social security systems and other social safety nets as they get richer. The one countercurrent is education. Some countries offer university education funded by taxpayers. Others charge as if education were a private good. There

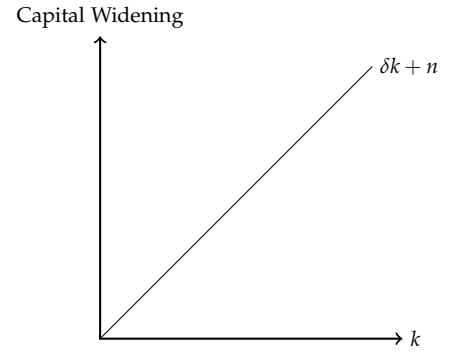


Figure 1: Plot of capital widening $(n + \delta)k$ as a function of k

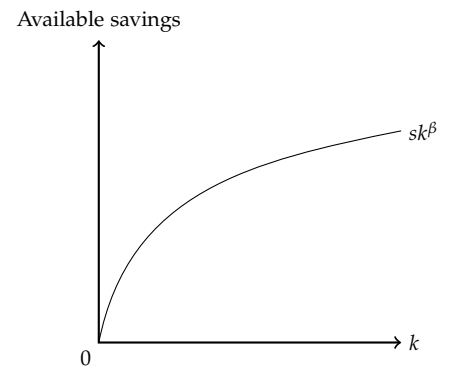


Figure 2: Plot of available savings per worker (sk^β) as a function of k

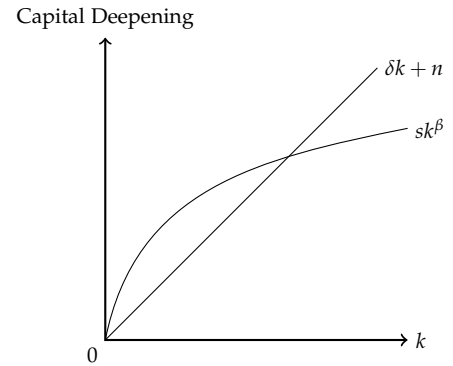


Figure 3: Plot of capital deepening (sk^β) as a function of k

does not seem to be convergence here.

Some practice problems

Example 1 An economy (a) has $A = 1, n = 0.03, s = 0.4, \delta = 0.03$. Solve for the steady-state level of $k_{ss} = \left(\frac{s}{n+\delta}\right)^2$, the capital per worker.¹⁰

Example 2 For economy (a) the initial capital-labor ratio is 4. Solve for $\delta k = sA\sqrt{k} - (n + \delta)k$.¹¹

Example 3 For economy (a) solve for the capital-labor ratio after one period.¹²

Example 4 For economy (b) $A = 1, n = 0.015, s = 0.2, \delta = 0.05$. Solve for the steady-state level of k .¹³

Example 5 For economy (b) let the level of k be 16, solve for δk .¹⁴

Example 6 For economy (b), the initial capital-labor ratio is 4. Solve for the capital-labor ratio after one period.¹⁵

Example 7 Compare the two economies (a) and (b). Is one more developed than the other? **Discussion:** not initially since they are both starting at the same place, with identical SAMs. In the steady state, economy (a) has a lot more capital per person, so it is probably a developing country, at least compared to (b). They both have the same square-root production function but the saving rate is higher in (a) relative to (b) while $n + \delta$ is approximately the same. Since the (a) saves more and the parameters that determine capital widening are similar, (a) will have a higher income per worker than (b) even though, at the moment, (a) is more productive. Eventually (a) will become much more productive than (b). Does this bring to mind any current examples? This may not be so realistic, however, since the population growth rate in (a), which is very high, will probably slow down with the demographic transition. On the other hand, the savings rate might decline as well. In this case, the two economies would *converge* to the same level of income. For that to happen exactly, the level of depreciation would also have to be the same. Note that the growth rate of the capital-labor ratio in the steady state is zero for both economies. From $\hat{K} - n = \hat{k}$ this implies that the rate of growth of the capital stock, \hat{K} is equal to the growth rate of labor, n . Initially, however, the growth rate of the capital-labor $\hat{k} = \frac{\Delta k}{k}$ is 14 percent in (a) and 4 percent in (b). This means that the growth rate of income per worker is 7 percent in (a) and only 2 percent in (b).

Example 8 In the SAM above, the wage rate is unity and the technological coefficient, $A = 1$. Solve for employment, GDP and income per worker.¹⁶

Example 9 Solve for the capital stock given a square-root production function.¹⁷

¹⁰ Solution: $k_{ss} = 44.44$. The steady-state level of income per worker is $y_{ss} = A\sqrt{k} = \left(\frac{s}{n+\delta}\right) = 6.67$. If half the working age population is employed, then the income per capita is 3.34.

¹¹ Solution: $\delta k = 0.560$.

¹² Solution: for period 2: $k_2 = 4.56$.

¹³ Solution: $k = 9.47$. The income per worker in the steady-state is $y_{ss} = 3.08$ and the if 2/3 of the working age population is in the labor force, income per capita is 2.05.

¹⁴ Solution: $\delta k = -0.24$.

¹⁵ Solution: $k = 4.14$.

¹⁶ Solution: Payments to labor divided by 1 is employment; here it is 250. $GDP = 500$. GDP per worker is then 2.

¹⁷ Solution: $y = A\sqrt{k}$ implies $y^2 = k = K/L$. This gives $K = Ly^2$. The capital stock is then $250(2^2) = 1000$ The initial value of $k = 4$.

Table 1: Social Accounting Matrix

| | Firms | HHs | Invest | Total |
|---------|-------|-----|--------|-------|
| Firms | | 400 | 100 | 500 |
| HHolds | 500 | | | 500 |
| profits | 250 | | | 250 |
| wages | 250 | | | 250 |
| Savings | | 100 | | 100 |
| Total | 500 | 500 | 100 | |

Source: made up numbers.

Example 10 Solve for the capital-labor ratio after one period in the SAM above, assuming $\delta = 0.025$, $n = 0.015$, $s = 0.3$ ¹⁸

Example 11 Solve for the rate of growth of the capital-labor ratio in the first period.¹⁹

Example 12 Solve for the steady-state k_{ss} and the steady-state level of income per worker, y_{ss} .²⁰

Excel file to calculate results.

¹⁸ Solution: $k_{t+1} = 4.44$.

¹⁹ Solution: $\Delta k/k = 0.11$.

²⁰ Solution: $k_{ss} = 56.25$ and $y_{ss} = 7.5$.