Some Arithmetic Deformation Theory

Taylor Dupuy

September 15, 2010
Prop

If \( R \) is a ring of characteristic \( p \) then the map \( F : x \mapsto x^p \) is a ring endomorphism.

Lift of Frobenius

A lift of the Frobenius on \( R \) is a map \( \sigma : R \to R \) such that

\[
\sigma(x) \equiv x^p \mod p.
\]
Absolute Frobenius

$k$ perfect of characteristic $p$. $X$ a smooth scheme defined over $k$,

**Absolute Frobenius**

Morphism of schemes

$$F : X \rightarrow X$$

1. Identity on topological space
2. $f \mapsto f^p$ on sheaf.
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Absolute Frobenius

Morphism of schemes

$$F : X \rightarrow X$$

1. Identity on topological space
2. $f \mapsto f^p$ on sheaf. We are changing the sheaf:

$$\mathcal{O}_X(U) \rightarrow F_*\mathcal{O}_X(U)$$
What is the Deligne-Illusie Class?

- Obstruction to lift of frobenius $\mod p^2$.
- Used in a paper by Deligne and Illusie in 1987 to give an algebraic proof of Kodaira Vanishing'

$$H^i(X, L \otimes \omega) = 0 \text{ for } i > 0$$
What is the Deligne-Illusie Class?

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$$H^i(X, L \otimes \omega) = 0 \text{ for } i > 0$$

(for example in Hartshorne $X$ is taken to be a nonsingular projective variety over $\mathbb{C}$)
Deligne and Illusie Class

The Deligne-Illusie Obstruction

$X \in H^1(X_{\text{reduction mod } p}, F^* T_X)$

Frobenius Tangent Bundle

$X_p = X \otimes \mathbb{R}/ \mathbb{R}/p \mathbb{R}$, perfect

Recall: $\text{char}(\mathbb{R}) = p = \Rightarrow \exists x \mapsto x^p$; the Frobenius always makes sense in characteristic $p$.

$\theta \in \Gamma(U, F^* T_X)$ means $\theta : O(U) \to O(U)$ with $\theta(fg) = \theta(f)g^p + f^p \theta(g)$

$\theta(f + g) = \theta(f) + \theta(g)$
The Deligne-Illusie Obstruction

\[ \text{DI}_X \in H^1 \left( \underbrace{X_p}_{\text{reduction mod } p}, \underbrace{F^*TX_p}_{\text{Frobenius Tangent Bundle}} \right) \]

\[ X_p = X \otimes_R R/pR, \ R/pR \text{ perfect} \]
The Deligne-Illusie Obstruction

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Recall:

- \( \text{char}(R) = p \implies \exists x \mapsto x^p \); the frobenius always makes sense in characteristic \( p \).
- \( \theta \in \Gamma(U, F^*TX_p) \) means \( \theta : \mathcal{O}(U) \to \mathcal{O}(U) \) with
  \[
  \begin{align*}
  \theta(fg) &= \theta(f)g^p + f^p\theta(g) \\
  \theta(f + g) &= \theta(f) + \theta(g)
  \end{align*}
  \]
Theorem \[ \text{DI} \cdot X = 0 \implies O \cdot X \text{ admits a lift of the Frobenius mod } p^2 \]

Lift: \( \sigma: O \cdot X \otimes_R R/p^2 R \to O \cdot X \otimes_R R/p^2 R \) such that \( \sigma(f) \equiv f \pmod{p} \).
Theorem

$\text{DI}_X = 0 \iff \mathcal{O}_X \text{ admits a lift of the Frobenius } \mod p^2$
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\[ \text{DI}_X = 0 \iff \mathcal{O}_X \text{ admits a lift of the frobenius } \mod p^2 \]

Lift: \[ \sigma : \mathcal{O}_X \otimes_R R/p^2 R \to \mathcal{O}_X \otimes_R R/p^2 R \text{ such that } \sigma(f) \equiv f^p \mod p. \]
Construction of $\text{DI}_X$

$X$ smooth scheme over $R$.
$R$ admits a lift of the Frobenius, and $R/pR$ a perfect field.

Cover $X$ by affine open sets $X_i$.

Lift of the Frobenius to $X_i$ (by smoothness).

We want to determine how these patch:

Prop $1 \ x \mapsto (\sigma_i(x) - \sigma_j(x)) p \mod p$

derivation of $F$.

$2 (\sigma_i - \sigma_j) p \mod p \in Z_1(X_p,F^*T X_p)$.

Actually a well-defined class.

PROBLEM: Lift this cocycle to characteristic zero.

Recipient Class? $\hat{\mathbb{Z}}$ ur $p(O,O)$ (Explain later)
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2. Lift of the Frobenius to $X_i$. 

We want to determine how these patch:

$\sigma_i(x) \mapsto (\sigma_i(x) - \sigma_j(x) p) \mod p$

Derivation of $F$.

$\sigma_i - \sigma_j p \mod p \in \mathbb{Z}^{1}(X^p, F^* T X^p)$.

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Prop

1. $x \mapsto \left(\frac{\sigma_i(x) - \sigma_j(x)}{p}\right) \mod p$ derivation of $F$. 

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Recipient Class? $\text{Mod}_{\hat{\mathbb{Z}}_p}^\text{ur}(\mathcal{O}, \mathcal{O})$ (Explain later)
Prop

\( R \) a ring

\[ \exists \sigma : R \to R \iff \exists \delta : R \to R \]

\( \delta \) a \( p \)-derivation
Prop

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\( \delta \) a \textit{p-derivation}

(\textit{Not Actually True})
Prop

$R$ a ring

$\exists \sigma : R \to R \iff \exists \delta : R \to R$

$\delta$ a $p$-derivation

(Not Actually True)
Prop

\[ R \text{ a ring} \]

\[ \exists \sigma : R \to R \iff \exists \delta : R \to R \]

\(\delta\) a \(p\)-derivation

(Not Actually True)

\[ R \text{ has } \sigma \iff R \text{ has } \delta \quad (1) \]
Prop

$R$ a ring

\[ \exists \sigma : R \to R \iff \exists \delta : R \to R \]

$\delta$ a $p$-derivation

(Not Actually True)

\[ R \text{ has } \sigma \iff R \text{ has } \delta \]

\[ R \text{ has } \sigma \implies R \text{ has } \delta, \]  \hspace{1cm} (1)
Prop

\( R \) a ring

\[ \exists \sigma : R \to R \iff \exists \delta : R \to R \]

\( \delta \) a \( p \)-derivation

(Not Actually True)

\( R \) has \( \sigma \)  \iff  \( R \) has \( \delta \)  \hspace{1cm} (1)

\( R \) has \( \sigma \)  \implies  \( R \) has \( \delta \), \hspace{1cm} When \( R \) is \( p \)-torsion free \hspace{1cm} (2)
What is a $p$-derivation?

Given a $\sigma$:

$$\delta(x) := \sigma(x) - x$$

"ratio of two zeros in characteristic $p$"

Given a $\sigma$:

$$\sigma(x) = x^p + p\delta(x)$$
What is a \(p\)-derivation?

Given a \(\sigma\):

\[
\delta(x) := \frac{\sigma(x) - x^p}{p}
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“ratio of two zeros in characteristic \(p\)”
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What is a $p$-derivation?

WARNING: $p$-derivations are nonlinear.

\[
\delta(xy) = \delta(x)y^p + x^p\delta(y) + p\delta(x)\delta(y)
\]

\[
\delta(x + y) = \delta(x) + \delta(y) + \frac{x^p + y^p - (x + y)^p}{p}
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(WARNING: not even linear mod $p$)
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**Example:**

\[
\delta\left(\frac{1}{x}\right) = \frac{1}{x^p} \frac{1}{1 + p\frac{\delta(x)}{x^p}}
\]
Reinterpretation of $\text{DI}_X$

DI revisited:

$$\sigma_i(x) - \sigma_j(x) = (x + \delta_i(x)) - (x + \delta_j(x))$$

Prop. Differences of $p$-derivations are derivations of the Frobenius when reduced mod $p$. They are $\hat{\mathbb{Z}}$-linear maps. As before: unreduced are they a cocycle in the sheaf defined by $U \mapsto \text{Nat}(\mathcal{O}|_U, \mathcal{O}|_U)$?
DI revisited:

\[ \frac{\sigma_i(x) - \sigma_j(x)}{p} \]
Reinterpretation of $DI_X$

DI revisited:

$$\frac{\sigma_i(x) - \sigma_j(x)}{p} = \frac{(x^p + p\delta_i(x)) - (x^p + p\delta_j(x))}{p}$$

Prop:

Differences of $p$-derivations are derivations of the frobenius when reduced mod $p$. They are $\hat{Z}$ linear maps. As before: unreduced are they a cocycle in the sheaf defined by $U : \rightarrow Nat(O|_U, O|_U)$.
Reinterpretation of $\text{DI}_X$

**DI revisited:**

\[
\frac{\sigma_i(x) - \sigma_j(x)}{p} = \frac{(x^p + p\delta_i(x)) - (x^p + p\delta_j(x))}{p} = \delta_i(x) - \delta_j(x)
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Reinterpretation of $\text{DI}_X$

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Prop

- *Differences of $p$-derivations are derivations of the frobenius when reduced mod $p$.*
Reinterpretation of $\text{DI}_X$

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Prop

- Differences of $p$-derivations are derivations of the frobenius when reduced mod $p$.
- They are $\mathbb{Z}_p^{ur}$ linear maps.
Reinterpretation of $\text{DI}_X$

$\text{DI}$ revisited:

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\frac{\sigma_i(x) - \sigma_j(x)}{p} = \frac{(x^p + p\delta_i(x)) - (x^p + p\delta_j(x))}{p} = \delta_i(x) - \delta_j(x)
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**Prop**

- *Differences of $p$-derivations are derivations of the frobenius when reduced mod $p$.*

- *They are $\mathbb{Z}_p^{ur}$ linear maps.*

As before: unreduced are they a cocycle in the sheaf defined by $U \mapsto \text{Nat}(\mathcal{O}|_U, \mathcal{O}|_U)$ “Sheaf Hom”?
Reynaud’s Theorem

Theorem

If $C$ is a smooth projective curve of genus $g \geq 2$ then $C$ does not admit a lift of the Frobenius for all primes $p$. (Say defined over $\mathbb{Z}$ then get $C$ over $\widehat{\mathbb{Z}}$ up by base extension).

Easier Examples:

Affine $X$. Always $g(\mathbb{P}^1) = 0$. Yes

Elliptic. Sometimes

Abelian. Sometimes
Reynaud’s Theorem

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Reynaud’s Theorem

**Theorem**

*If* $C$ *is a smooth projective curve of genus* $g \geq 2$ *then* $C$ *does not admit a lift of the frobenius for all primes* $p$.

(Say defined over $\mathbb{Z}$ then get $C$ over $\mathbb{Z}_p^{ur}$ by base extension).

Easier Examples:

- Affine $X$. Always
- $g(\mathbb{P}^1) = 0$. yes
- $E$. sometimes
- $A$. sometimes
$X/R$ is $p$-torsion free $R/pR$ perfect

$\sigma : R \to R$ lift of the frobenius.

**QUESTION:** When does $X$ admit a lift of the absolute frobenius agreeing with $\sigma$ on $R$?

**PARTIAL ANSWER:** Necessary condition: $\text{DI}_X = 0$. 

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Some Arithmetic Deformation Theory
Summary

\( X/R \) smooth scheme

QUESTION: When does \( X \) admit a lift of the absolute frobenius agreeing with \( \sigma \) on \( R \)?

PARTIAL ANSWER: Necessary condition: \( DI_X = 0 \).
Summary

\( X/R \) smooth scheme
\( R \) is \( p \)-torsion free

**Question:** When does \( X \) admit a lift of the absolute Frobenius agreeing with \( \sigma \) on \( R \)?

**Partial Answer:** Necessary condition: \( \text{DI}_X = 0 \).
Summary

\(X/R\) smooth scheme
\(R\) is \(p\)-torsion free
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QUESTION:
When does \(X\) admit a lift of the absolute frobenius agreeing with \(\sigma\) on \(R\)?

PARTIAL ANSWER:
Necessary condition:
\(DI_X = 0\).
$X/R$ smooth scheme
$R$ is $p$-torsion free
$R/pR$ perfect
$\sigma : R \to R$ lift of the frobenius.
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Necessary condition: \( \text{DI}_X = 0 \).
HOW IS $D\mathbb{I}_X$ A DEFORMATION CLASS?
A Similar Construction using Derivations

Let $X/R$ be a smooth scheme and where $R$ has a derivation $\delta$. Cover $X$ by affine open subsets $X_i \sim \text{Spec}(O(X_i))$. $X_i \subset X \rightarrow \text{Spec}(R)$ still smooth. The derivation $\delta: R \rightarrow R$ lifts (nonuniquely) to a derivation $\delta_i: O(X_i) \rightarrow O(X_i)$. The differences $\delta_i - \delta_j$ give an $R$-linear derivation. These give a cohomology class $[\delta_i - \delta_j] \in H^1(X, TX)$. 

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Some Arithmetic Deformation Theory
A Similar Construction using Derivations

FUNCTION FIELD SETTING

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Let $X/R$ be a smooth scheme and where $R$. has a derivation $\delta$. Cover $X$ by affine open subsets $X_i \cong \text{Spec}(O(X_i))$. $X_i \subset X \rightarrow \pi \text{Spec}(R)$ still smooth. The derivation $\delta: R \rightarrow R$ lifts (nonuniquely) to a derivation $\delta_i: O(X_i) \rightarrow O(X_i)$. The differences $\delta_i - \delta_j$ give an $R$-linear derivation. These give a cohomology class $\left[\delta_i - \delta_j\right] \in H^1(X, TX)$. 

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Some Arithmetic Deformation Theory
FUNCTION FIELD SETTING

Let \( X/R \) be a smooth scheme and where \( R \) has a derivation \( \delta \).

Cover \( X \) by affine open subsets \( X_i \), where:

\[ X_i \cong \text{Spec}(O(X_i)) \]

\( X_i \subset X \rightarrow \pi \text{Spec}(R) \) still smooth.

The derivation \( \delta: R \rightarrow R \) lifts (nonuniquely) to a derivation \( \delta_i: O(X_i) \rightarrow O(X_i) \).

The differences \( \delta_i - \delta_j \) give an \( R \)-linear derivation. These give a cohomology class \( [\delta_i - \delta_j] \in H^1(X, \mathcal{T}_X) \).
A Similar Construction using Derivations

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- Cover $X$ by affine open subsets $X_i \cong \text{Spec}(\mathcal{O}(X_i))$.

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Let $X/R$ be a smooth scheme and where $R$. has a derivation $\delta$.

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$X_i \subset X \longrightarrow \pi \text{Spec}(R)$ still smooth.
FUNCTION FIELD SETTING

- Let $X/R$ be a smooth scheme and where $R$. has a derivation $\delta$.
- Cover $X$ by affine open subsets $X_i \cong \text{Spec}(\mathcal{O}(X_i))$.
- $X_i \subset X \overset{\pi}{\longrightarrow} \text{Spec}(R)$ still smooth.
- The derivation $\delta : R \rightarrow R$ lifts (nonuniquely) to a derivation $\delta_i : \mathcal{O}(X_i) \rightarrow \mathcal{O}(X_i)$.
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Let $X/R$ be a smooth scheme and where $R$. has a derivation $\delta$.

Cover $X$ by affine open subsets $X_i \cong \text{Spec} (\mathcal{O}(X_i))$.

$X_i \subset X \longrightarrow \pi \text{Spec}(R)$ still smooth.

The derivation $\delta : R \rightarrow R$ lifts (nonuniquely) to a derivation $\delta_i : \mathcal{O}(X_i) \rightarrow \mathcal{O}(X_i)$

The differences $\delta_i - \delta_j$ give an $R$-linear derivation. These give a cohomology class

$$[\delta_i - \delta_j] \in H^1(X, TX).$$
A Cocycle Construction Using Lifts of Derivations

Theorem

The following are equivalent

1. $\text{KS}(\delta) = 0$

2. $J_1(X) \sim TX$

3. $X$ descends to $R$ where $\delta = \{ r \in R : \delta(r) = 0 \}$ meaning $X \sim X_0 \otimes R$ where $X_0$ is some scheme defined over $R$.

When $X$ is a variety, $R = \mathbb{C}(x)$ defining equations have coefficients in $\mathbb{C}$. 

Taylor Dupuy
Some Arithmetic Deformation Theory
Kodaira-Spencer Map:

\[ KS_{X/R} : \text{Der}(R, R) \to H^1(X, TX). \]

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Some Nonsense:

Vague (and popular) Analogy is employed:

$$KS = 0 \iff DI = 0$$

Interpretation: When trying to modify theorems in the function field setting, view a lift of the frobenius which keeps the topology fixed as "descent to $$F_1$$".
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WHERE IS THE DEFORMATION THEORY?
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- $K_{S}^{\text{ext}}$ denote the class before.
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- $KS^{\text{ext}}$ denote the class before.
- We are now going to define $KS^{\text{ext}}$. 
Let $\mathcal{X}$ over $S$ be a smooth and flat. We view it as a family of varieties

$$\mathcal{X}_P := \mathcal{X} \otimes_S \kappa_S(P)$$
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Important Examples

\[\begin{array}{ccc}
\mathcal{E} & \mathcal{C}_g & \mathcal{A} \\
\downarrow & \downarrow & \downarrow \\
\mathbb{A}^1 & M_g^n & N_{1,g}^{(n)}
\end{array}\]
Given a direction in the moduli we can define a cocycle
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\[ \text{KS}^{\text{ext}} : T_P S \to H^1(X_P, T_{X_P}) \text{.} \]
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Given a direction in the moduli we can define a cocycle (and well defined cohomology class):

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1. Let \( \delta_P \in T_P S \)
2. Fatten to some open affine neighborhood \( U \) of \( S \) which contains \( P \).
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2. Fatten to some open affine neighborhood \( U \) of \( S \) which contains \( P \). Call the extended derivation \( \delta \).
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3. Cover \( \mathcal{X} \) by affine \( \mathcal{X}_i \), get \( \delta_i \)
4. \( \theta_{ij} := (\delta_i - \delta_j) \) are completely vertical (they vanish under \( \pi_* = d\pi \) and hence restrict to cocycles on the fiber)

\[ \text{KS}^{\text{ext}}(\delta_P) := [\delta_i - \delta_j] \in H^1(\mathcal{X}_P, T\mathcal{X}_P). \]
Link Between Internal and External

\[
\begin{array}{c}
\text{Deformation + Derivation on Base:}
\end{array}
\]

\[
\begin{array}{c}
\downarrow \\
\downarrow \\
\leftarrow \\
\leftarrow \\
\downarrow \\
\downarrow \\
\end{array}
\]

\[
\begin{array}{c}
\text{Spec(} K \text{)} \\
\leftarrow \\
\leftarrow \\
\text{model of} \\
X \\
\delta \\
O(S) \rightarrow O(S)
\end{array}
\]

Let \( \kappa_S \) (generic pt of \( S \)) = \( K \).
$X$ be defined over a function field $K$
Link Between Internal and External

\( X \) be defined over a function field \( K \)

Deformation + Derivation on Base:

\[
\begin{array}{c}
\mathcal{X} \leftarrow X \\
\downarrow \quad \downarrow
\end{array}
\]

\[
\begin{array}{c}
S \leftarrow \text{Spec}(K)
\end{array}
\]

model of \( X \)

\( \delta : \mathcal{O}(S) \rightarrow \mathcal{O}(S) \)
\( X \) be defined over a function field \( K \)

Deformation + Derivation on Base:

\[
\xymatrix{ \mathfrak{X} & X \ar[l] \ar[d] & \delta : \mathcal{O}(S) \to \mathcal{O}(S) \\
S & \text{Spec}(K) \ar[l] } \]

\( \mathfrak{X} \) is a model of \( X \),

Let \( \kappa_S(\text{generic pt of } S) = K \).
Construct $\eta \in H^1(X,T_X)$ by lifting the $\delta$ on affine open sets as we have been doing previously. Specializing $\eta$ at the generic points gives $KS_{\text{ext}} \eta \otimes \kappa$ (generic) $= KS_{\text{ext}}$. Specializing $\eta$ at the closed points gives the external construction $\eta \otimes \kappa$ (closed point $P$) $= KS_{\text{ext}}$. Some Arithmetic Deformation Theory
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$$\eta \otimes \kappa(\text{generic}) = \text{KS}^{\text{ext}}$$
Construct \( \eta \in H^1(\mathcal{X}, T\mathcal{X}) \) by lifting the \( \delta \) on affine open sets as we have been doing previously.

Specializing \( \eta \) at the generic points gives \( \text{KS}^{\text{ext}} \)

\[
\eta \otimes \kappa(\text{generic}) = \text{KS}^{\text{ext}}
\]

Specializing \( \eta \) at the closed points gives the external construction

\[
\eta \otimes \kappa(\text{closed point } P) = \text{KS}^{\text{ext}}
\]
Remark: There is a third way to get the map using the relative tangent sequence. Specializing that map gives different versions of this map.
More than Just an Analogy

Theorem (DI can detect deformation theoretic information)

If $A$ is an abelian variety then

$$F^*KS^{\text{ext}} = \text{DI}$$

PROBLEM: Find external constructions in the $p$-derivation setting. And relate them to known classes.
Infinitesimal Deformations

The Kodaira Spencer map for any deformation factors through infinitesimal deformations.

\[
\begin{array}{c}
\text{Spec}(k) \\
\end{array}
\begin{array}{c}
\xrightarrow{\epsilon} \\
\end{array}
\begin{array}{c}
\text{Spec}(k[\epsilon]) \\
\end{array}
\]

Lemma

For all deformations families of \( X \) and choice tangent vector at a point who fiber is \( X \), there exists an infinitesimal deformation that gives rise to the same cohomology class.
"The Kodaira Spencer map for any deformation factors through infinitesimal deformations."

\[ X^\epsilon \leftarrow X \]
\[ \downarrow \quad \downarrow \]
\[ \text{Spec}(k[\epsilon]) \leftarrow \text{Spec}(k) \]

**Lemma**

*For all deformations families of $X$ and choice tangent vector at a point who fiber is $X$, there exists an infinitesimal deformation that gives rise to the same cohomology class.*
Define a functor

$$\text{Def}_X : \{ \text{Local Artin Rings} \} \rightarrow \text{Sets}$$
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\[ \text{Def}_X : \{ \text{Local Artin Rings} \} \rightarrow \text{Sets} \]

where

\[ \text{Def}_X(A) = \{ \mathfrak{x} \rightarrow \text{Spec}(A), \mathfrak{x} \otimes \sim \} \]

Prop

\[ \text{Def}_X(K[\epsilon]) \leftrightarrow H^1(X, TX) \]
Factoring properties

Differentiation

\[ R \longrightarrow D_1(R) \cong R[\epsilon] \]

Wittdifferentiation

\[ R \longrightarrow W_1(R) \]

These allow lifting of derivations and \( p \)-derivations for smooth maps.
Wittfinitesimal Deformations

We can construct a map similar to the one for wittfinitesimals in the case when $\text{char}(k) = p$

$$\text{Def}(W_p(k)) \rightarrow H^1(X, F^*TX).$$

**PROBLEM:**
Understand the wittfinitesimal deformations. Understand wittfinitesimal versions of the torelli map.
Big Class

\[ B_X \in H^1(X\hat{\cdot}, \text{Aut}(\mathbb{A}^1)\hat{\cdot}) \]

(that is terrible to \TeX)

Let \( X \) be a smooth scheme over \( W_p^\infty(\overline{\mathbb{F}}_p) = \left( \mathbb{Z}_p^{ur} \right) \) has unique lift of frob
Big Class

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Let \( X \) be a smooth scheme over \( W_p^\infty(\overline{\mathbb{F}}_p) = \mathbb{Z}_p^{ur} \).

has unique lift of frob

Define cohomology class in the sheaf \( \text{Aut}(\mathbb{A}^1)^\hat{p} \) via the local trivializations of the jet space.
The Big Class

[Jet Rings and Globalizing]
The Big Nasty Class

What the sheaf looks like (hat’s omitted):

\[ \Gamma(U, \text{Aut}(\mathbb{A}^1)^\hat{p}) = \left\{ \varphi : U \times \mathbb{A}^1 \rightarrow U \times \mathbb{A}^1 \right\} \]
The cocycle is induced by transition maps between trivializations. Let $X_i \subset X$ be trivializing sets. If $X$ is smooth over $\widehat{\mathbb{Z}}_{ur}$ of relative dimension $d$ then

$$J^n(X_i) \sim X_i \times \mathbb{A}^{dn}$$
The cocycle is induced by transition maps between trivializations
Let $X_i \subset X$ be trivializing sets
If $X$ is smooth over $\widehat{\mathbb{Z}/p}$ of relative dimension $d$ then

\[ J^n(X_i) \to \sim X_i \times \mathbb{A}^{dn} \]

*Characteristic Zero
What does the $B$ actually look like?
What does the $B$ actually look like? Let $X = C$ a curve and $n = 1$.

$$\mathcal{O}(\hat{C}_{i} \times \hat{A}^1) = \mathcal{O}(C_{i})[\hat{x}]^{\hat{p}}$$

consisting of restricted powerseries $\sum_{j=0}^{\infty} f_{j} \hat{x}^{j}$ satisfying $|f_{j}|_{p} \to 0$ as $p \to \infty$.

***Transition maps happen by plugging in restricted power series.
Lemma

Let $R$ be a ring of characteristic $p$. Then

$$\text{Aut}_R(\mathbb{R}[x]) \cong \text{AL}_1(R).$$

The automorphisms look like

$$f(x) \mapsto f(ax + b)$$

where $a \in R^\times$ and $b \in R$. 
When we reduce the cocycle in $\text{Aut}(\hat{A}^1) \otimes_{\mathbb{Z}_p} \overline{\mathbb{F}}_p \cong AL_1$.

$$AL_1 = \mathbb{G}_m \ltimes \mathbb{G}_a$$

$$\beta_{ij} = c_{ij} \dot{x} + d_{ij}$$

gives $[F^*TX] \in H^1(C, O^\times)$ gives DI
PROBLEM
We want to show $B(X)$ is nontrivial.