Difference Quotients, Derivatives, and Data through Modeling with Slime

Richard M. Single, Department of Mathematics, St. Olaf College, Northfield, Minnesota

Contact Information:

Richard M. Single
1600 East Third Ave. #2612
San Mateo, CA 94401
(650) 344-6079
single@stolaf.edu

Abstract

In this article, I present an experiment that can be conducted in a calculus class to investigate the difference quotient and the derivative, using mathematical modeling with student-collected data. I also discuss an extension to the experiment through which students can discover the meaning of parameter estimation in a mathematical model.

KEY WORDS: Mathematical Modeling, Data Analysis, Difference Quotient

1 Introduction

The importance of the connection between the difference quotient and the derivative is easily missed by beginning calculus students. The difference quotient is too often viewed as a second class citizen, which seems appropriate only when the "real function" (an algebraic representation describing the relationship between two variables) is not known. Modeling a physical phenomenon or system can provide a concrete and hands-on connection between these ideas. In this way, students are able to investigate the relationship between two variables and their rates of change. This type of investigation is even more compelling for students when they derive the relationships using their own data. For many students this can be their first introduction to a mathematical model. Computations and plots for the experiment can be done using either a spreadsheet or a graphing calculator that can plot the graph of a function over a scatterplot of data, such as the TI-83, TI-85, or TI-92.

The typical models presented in a calculus course provide little opportunity for students to investigate and question how representative they are of the true situation (see [1]). In this experiment, along with investigating mathematical properties of their models, students can use the data to examine the strengths and weaknesses of their models. Students work in groups to model the relationship between the volume of a blob of slime (see appendix for recipe), v, and the radius of the slime circle, r, after the slime has spread out. They also experimentally determine the degree to which a change in the volume of slime affects the radius of the resulting circle.

In addition, the experiment can be extended to demonstrate what it means to estimate a parameter in a model. This extension of the experiment can be used to highlight the interplay between mathematics and statistics, especially in the area of modeling (see [2] for some discussion of the benefits of integrating statistical ideas into mathematics courses). The most important aspects of the experiment are that it reinforces key concepts, uses student generated data, is memorable, and is fun. Students remember the “slime experiment” throughout the duration of the course and beyond. More importantly, they remember the concepts as well.

2 The Experiment

The experiment begins with students empirically determining the relationship between the volume of a blob of slime and the radius of the slime circle after the slime has spread out. To do so, several volumes of slime are measured out over the range of 1/4 teaspoon to 6 teaspoons (6 $tsp = 2\ Tbs$ $= 30{,}000\ mm^3$). Each blob is formed roughly into the shape of a sphere and left to settle on a transparency sheet. The transparency sheet serves two purposes. First, a thin ruler can be slid underneath to facilitate measuring. Second, although the slime is non-toxic,it can cause damage to a finished surface.

The diameter, in mm, of each puddle of slime is measured after it has finished settling, which takes roughly eight to ten minutes depending on the size of the original blob. If the puddle is not sufficiently round, I suggest to the students that they take several measurements along different axes and average them. Measurement issues such as these provide excellent discussion topics during the experiment. An estimate of the height of the slime puddle, h, is needed for later modeling.

Radius measurements, derived from the diameter because the diameter is easier to measure, can be entered in a table and then plotted against the volume measurements. Sample class data are given in the first four columns of Table 1. All measurements were taken after the slime had finished settling. The eight data points are plotted in Figure 1. A discussion of which is the dependent variable and which is the independent variable can help motivate the second step of the modeling procedure in which the radius is written as a function of the volume of a geometric solid. The plot of the observed data points in Figure 1 can be used to address questions about the nonlinearity of the radius-volume relationship. Does the rate of change of radius with respect to volume appear to be constant? Where is this rate of change greatest or least?

Let r_1 denote the radius of the settled slime puddle for the smallest volume, v_1, and r_n denote the value for the largest volume, v_n. The difference quotient, computed as $(r_{i+1} - r_i)/(v_{i+1} - v_i)$, for $i = 1, 2, ..., n-1$, can be entered in a table and plotted against the volume measurements. The last column of Table 1 contains the computed values and a plot of the seven data points for the sample class data is shown in Figure 2. Students see that their computed values for the difference quotient agree with the observations made from the plotted points in Figure 1. The questions about the nonlinear nature of their observed radius-volume relationship help to add context for students when they derive nonlinear functions for this relationship in the modeling portion of the experiment.

3 Modeling

The amount of direction given at this point as well as the number and depth of assessment questions will depend on the level of the class and whether or not the modeling will be done in class or as a homework/lab.

The puddle of slime can be modeled as a cylinder, a hemisphere, or a combination of the two. Students find it eye-opening that there is no one correct model. However, they realize that some models are better than others. For example, in this experiment a "cylinder model" better describes the observed data than a "hemisphere model." This step helps the students realize that the data are used not only to create a model, but also to criticize a proposed model.

The relationship between the radius and volume of the chosen solid should be written with the radius as a function of the volume.For the "cylinder model" the relationship is $r = (v/(h\pi))^{1/2}$. In order to proceed, an estimate of the height of the cylinder is needed. This height is difficult to measure because it is a small value and the measurement is taken perpendicular to the table or desk. For these reasons, I encourage each group of students to take several estimates of the height and average them. The estimates from different groups can be pooled if it is desired to have each group work with the same height measurement. In the discussion that follows, I use the median of the classroom estimates, $h = 5\ mm$.

At this point I introduce the idea of modeling assumptions. Using a "cylinder model" the puddle of slime is assumed to have a uniform diameter and height for the purpose of simplifying the model. In reality, this is not true. Students come to realize that the degree to which these assumptions are true affects how well the model describes the observed data.

The next step allows the students to see how well their theoretical model describes their observed data. I have the students graph the relationship between the radius and volume for the theoretical model on the same set of axes as the observed data. The smooth curve in Figure 1 shows a graph of $r = (v/(5\pi))^{1/2}$, a "cylinder model" with a height of 5 mm, superimposed over the plot of the observed data points. Using their model the students can now roughly quantify the nonlinear relationship between the radius and the volume of the puddle of slime.

Finally, the students can compute the derivative of the function used to model the radius-volume relationship. Again, plotting the derivative (dr/dv) against the volume measurements on the same axes as the difference quotient gives a useful visual comparison (see Figure 2). This highlights the connection between the difference quotient and the derivative in an applied setting. The difference quotient describes the observed rate of change of radius with respect to volume for their experiment. The derivative of their modeled function gives them further insight into this rate of change. The smooth curve in Figure 2 shows a graph of $dr/dv = (1/(20\pi v))^{1/2}$, the derivative of the function used in the "cylinder model," superimposed over the plot of the data points representing the observed difference quotient.

4 Extensions

One extension for this exploration is an assessment of how well the model describes the observed data. This can be done in several ways. The model error, e, can be defined as the difference between the observed radius, r_{obs}, and the predicted radius, r_{pred}. That is, $e = r_{obs} - r_{pred}$. Each group can investigate these error values individually or the results for the entire class can be pooled in order to discuss more general properties of the errors for a specific model. Questions about where the model does a better or worse job of describing the observed data can lead students to consider strengths and weaknesses of their model. For example, in Figure 1, why does the model predict a much higher radius than observed for the largest volumes? This brings up the estimate of the height of the slime puddle as a weakness of the "cylinder model."

Another extension is to consider the role of a parameter in a model. For example, in the case of the "cylinder model," the height of the cylinder, h, can be considered as a parameter in the model $r = (v/(h\pi))^{1/2}$, since the height of the slime puddle is difficult to measure. Does the choice of $h = 5\ mm$ lead to the best performance for the model? How does changing the value of h affect the model errors? What value of h leads to the least model error or sum of squared model errors? These questions address some of the fundamental ideas of point estimation in statistics.

The easiest way to see how a change in the value of the parameter h affects the degree of fit for the theoretical model to the observed data is to superimpose several graphs of functions describing the predicted radius, using different values of h, over the observed data. Each group can visually compare graphs for several values of h and determine which value provides the best fit. The students can see that values of h that are near 5 mm provide the best fit. Figure 3 displays graphs of the function describing the cylinder model with h set to 4.5, 5.0, and 5.5 mm. Asking the students to describe how they made their decision provides a good introduction to the idea of minimizing model errors.

Since the model errors can be either negative or positive, a non-negative function, such as the square or absolute value, of the model errors is used to numerically compare the fit for different values of the parameter h. Table 2 contains squared model errors for the three different values of h

considered above. The smallest of these three values occurs at $h = 5.0\ mm$. This agrees with the graphical investigation using Figure 3 described above. To examine this further, Table 3 contains the sum of squared model errors for values of h in the range of 4.5 to 5.5 mm at increments of 0.1 mm. Figure 4 displays a graph of these values versus the value of h used in the model. For the observed data a value of $h = 5.2\ mm$ in the model leads to the smallest sum of squared model errors. Thus, using this as our criterion, we would choose $h = 5.2\ mm$ as the value of the height parameter in our model.

5 Conclusion

In-class experiments of the type described in this article serve several purposes. Among these are the reinforcement of key concepts and the generation of a high level of student interest driven by a sense of ownership of the results of the experiment. It is remarkable how many more students are engaged and actually recall (the concepts, not just the slime) when you add the phrase "as we observed in the slime experiment." In this experiment, the difference quotient describes the rate of change of radius with respect to volume for their observed data. The students have the opportunity to view the derivative, derived from their modeled function, as a tool for helping them to understand and describe the observed relationship.

Through the "slime experiment" students learn important lessons about mathematical modeling with real data. They learn that observed data plays a role in creating, critiquing, and refining a model. They also see that a mathematical model does not necessarily describe a phenomenon exactly. Assumptions are often made to simplify the model and its interpretation. Students learn to view a model as a tool for understanding a process. The following statement, attributed to statistician George Box, describes the situation well, "All models are wrong, but some are useful."

6 Appendix

Slime Recipe (courtesy of George Cobb):

4 oz. Elmer's glue

1 tsp. laundry borax

$1\frac{1}{4}$ cup water

Food coloring

Dissolve the laundry borax in one cup of water. Combine the food coloring, glue, and 1/4 cup of water in a large bowl. Stir in the borax solution. The mixture will congeal immediately. Drain off the excess water and store the slime in a sealed container. The slime will last for at least a week in the refrigerator, but will lose moisture with time and handling. One batch makes roughly 45 teaspoons.

7 References

1. Cobb, G.W., and Moore, D.S. (1997) Mathematics, Statistics, and Teaching, *Amer. Math. Monthly* **104**, 801-823.

2. Gordon, F.S, and Gordon, S.P. (1994) Combining Data Analysis and Probability in Precalculus, *Primus* **4**, No. 2, 109-116.

3. Ostebee, A., and Zorn, P. (1997) *Calculus from Graphical, Numerical, and Symbolic Points of View.* Philadelphia, PA: Saunders College Publishing Co.

8 Acknowledgements

I would like to thank George Cobb for the slime recipe and Martha Wallace and Peg Boyle Single for their help in refining the experiment.

Table 1. Measured diameter and radius data along with calculated difference quotient

Volume		Diameter	Radius	Difference Quotient
tsp	mm^3	mm	mm	mm/mm^3
0.25	1250	18.5	9.25	0.009
0.5	2500	29.75	14.875	0.0055
1.0	5000	43.5	21.75	0.0019
2.0	10000	53.0	26.5	0.0014
3.0	15000	60.0	30.0	0.0022
4.0	20000	71.0	35.5	0.001
5.0	25000	76.0	38.0	0.0009
6.0	30000	80.5	40.25	–

Table 2. Computation of sum of squared model errors for three values of the height parameter

		$h = 4.5$		$h = 5.0$		$h = 5.5$	
Volume	r_{obs}	r_{pred}	$(r_{pred} - r_{obs})^2$	r_{pred}	$(r_{pred} - r_{obs})^2$	r_{pred}	$(r_{pred} - r_{obs})^2$
1250	9.25	9.40	0.02	8.92	0.11	8.51	0.55
2500	14.875	13.30	2.49	12.62	5.10	12.03	8.10
5000	21.75	18.81	8.67	17.84	15.28	17.01	22.46
10000	26.5	26.60	0.01	25.23	1.61	24.06	5.97
15000	30.0	32.57	6.62	30.90	0.81	29.46	0.29
20000	35.5	37.61	4.46	35.68	0.03	34.02	2.18
25000	38.0	42.05	16.42	39.89	3.59	38.04	0.00
30000	40.25	46.07	33.82	43.70	11.92	41.67	2.01
–	–	Sum	72.52	Sum	38.45	Sum	41.57

Table 3. Sum of squared model errors

height	Sum of Squared Model Errors
4.5*	72.52
4.6	61.87
4.7	53.31
4.8	46.66
4.9	41.76
5.0*	38.45
5.1	36.61
5.2	36.11
5.3	36.83
5.4	38.68
5.5*	41.57

* Detail for these entries appears in Table 2

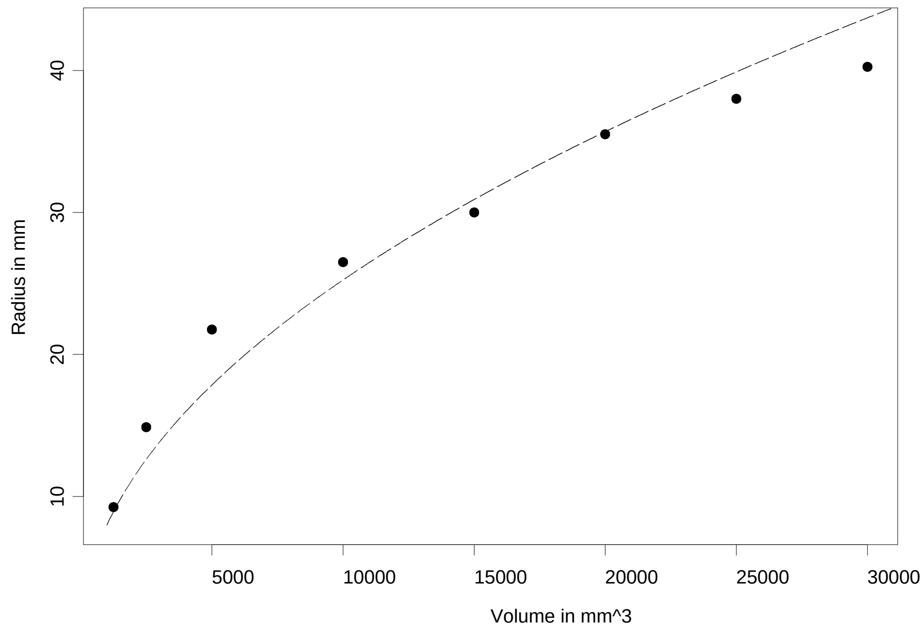

Figure 1: Modeled radius-volume relationship graphed over the observed data

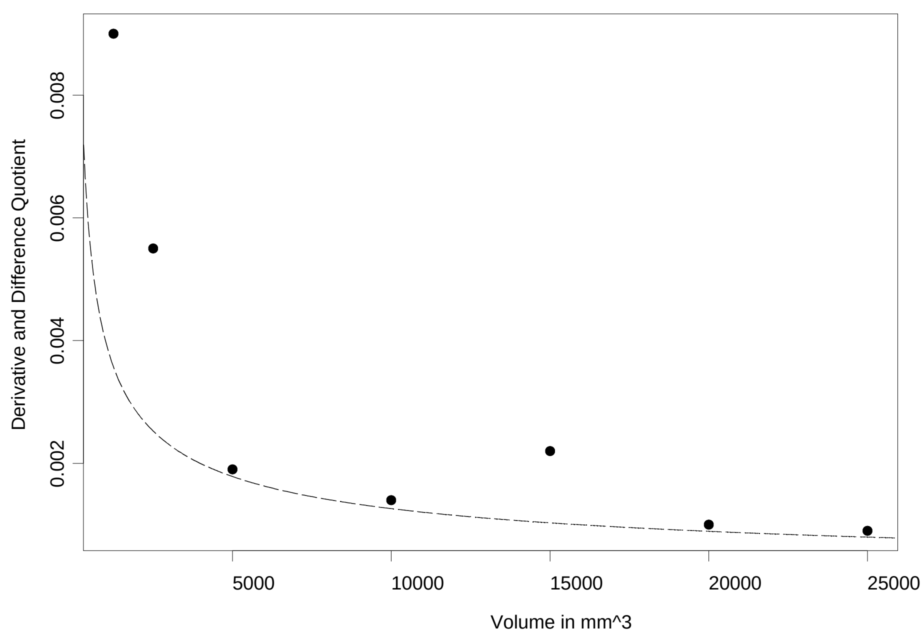

Figure 2: Derivative of the modeled radius-volume relationship graphed over the observed difference quotient

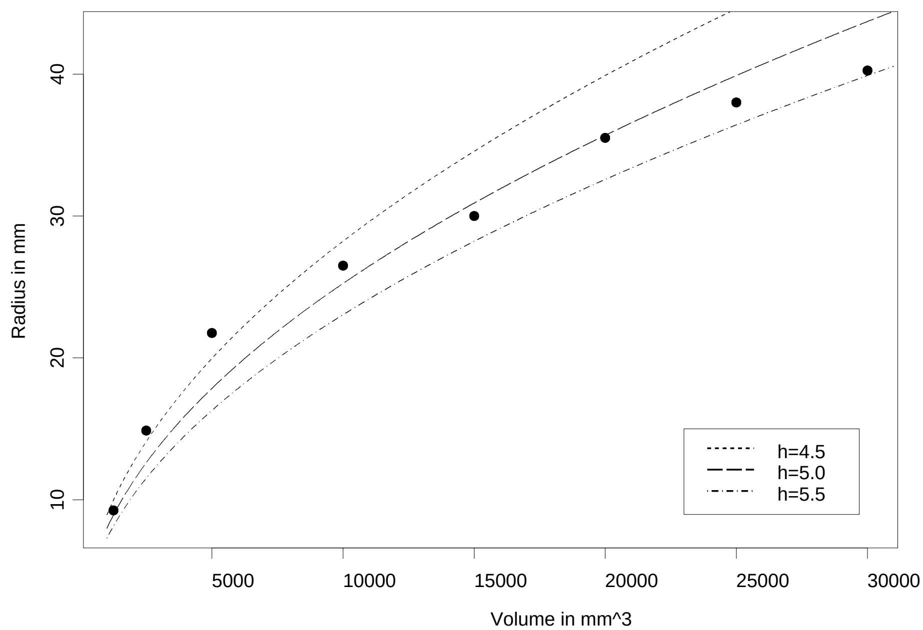

Figure 3: Graphs of the modeled radius-volume relationship for $h = 4.5$, $h = 5.0$, and $h = 5.5$ mm

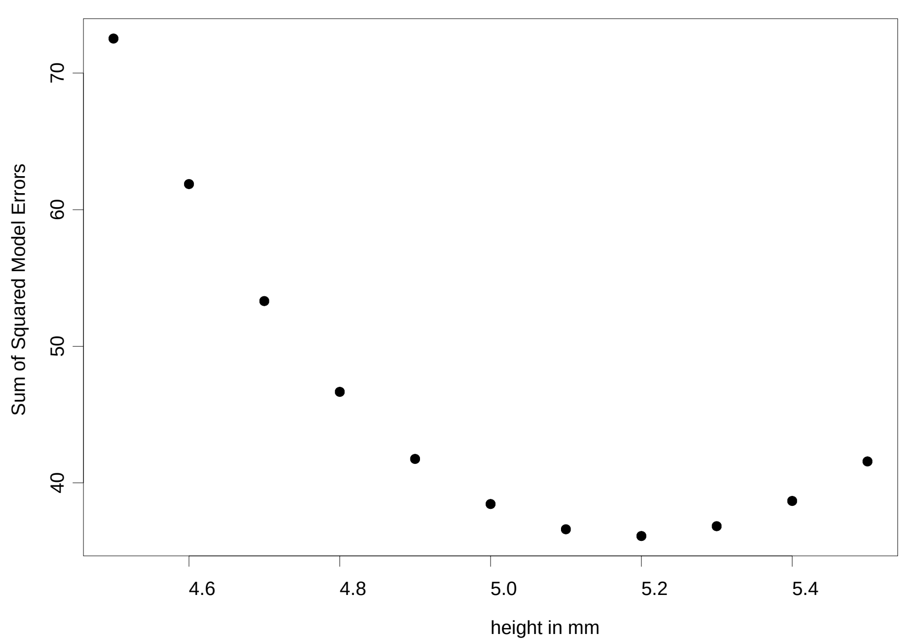

Figure 4: Plot of the sum of squared model errors versus the height parameter