Appendix A

The parameters of the income functions verify the following conditions. Recall that $z-b$ represent age. In the data, the labor income of the low income earners is lower than the labor income of the high income earners. Therefore:

$$\tilde{N}^l(b,t) < \tilde{N}^h(b,t)$$

and

$$\left[a^l e^{\alpha^l(t-b)} - e^{\beta^l(t-b)}\right] E^l(t) N^l(t) < \left[a^h e^{\alpha^h(t-b)} - e^{\beta^h(t-b)}\right] E^h(t) N^h(t) \text{for } z-b < n_{old}$$

As a result of the functional forms, labor income decreases and can even be negative for very old ages (larger than n_{old}). However, this is not a problem to represent the aggregate behavior of the economy. The parameters of the functional forms are chosen such that the integrals representing aggregate human wealth converge. For very old ages, there is only an infinitesimal number of agents who are still alive in the model and they do not matter in the aggregate with proper values of the model's parameters.

The labor income age profile of the low income earners and the high income earners is hump shaped. Therefore, there is a maximum for age $z-b = n_{\max}$ described by:

$$\frac{\partial}{\partial(z-b)}(a_l e^{\alpha_l(z-b)} - e^{\beta_l(z-b)}) = 0 \text{ for } z-b = n_{\max}$$

$$\frac{\partial^2}{\partial(z-b)}(a_h e^{\alpha_h(z-b)} - e^{\beta_h(z-b)}) < 0 \text{ for } z-b = n_{\max}$$

Appendix B

$$H(t) = \int_{-\infty}^{t} \frac{1}{\lambda_l + \lambda_h} e^{-\frac{1}{\lambda_h}(t-b)} H(b,t)\, db$$

with:

$$H(b,t) = \int_{t}^{+\infty} e^{-\int_t^z \left[\hat{r}(v) + \frac{1}{\lambda_h}\right] dv} \hat{\omega}^h(z) E^h(b,z) N^h(b,z)\, dz$$

Using the expression for individual human wealth, aggregate human wealth is:

$$H(t) = \int_{-\infty}^{t} \frac{1}{\lambda_l + \lambda_h} e^{-\frac{1}{\lambda_h}(t-b)} \left\{ \int_{t}^{+\infty} e^{-\int_t^z \left[\hat{r}(v) + \frac{1}{\lambda_h}\right] dv} \hat{\omega}^h(z) \left[a_h \exp^{\alpha_h(z-b)} - \exp^{\beta_h(z-b)}\right] \tilde{N}^h(z)\, dz \right\} db$$

I reorganize the integrals and split human wealth into two components ($H_\alpha(t)$ and $H_\beta(t)$). They do not have economic significance but it is necessary to reorganize the integral and solve the model.

$$H(t) = H_\alpha(t) - H_\beta(t)$$

where $H_\alpha(t)$ and $H_\beta(t)$ are:

$$H_\alpha(t) = \int_{-\infty}^{t} a_h \frac{1}{\lambda_l + \lambda_h} e^{-\left(\frac{1}{\lambda_h} - \alpha_h\right)(t-b)} \left\{ \int_{t}^{+\infty} e^{-\int_t^z \left[\hat{r}(v) + \frac{1}{\lambda_h} - \alpha_h\right] dv} \hat{\omega}^h(z) \tilde{N}^h(z)\, dz \right\} db$$

$$H_\beta(t) = \int_{-\infty}^{t} \frac{1}{\lambda_l + \lambda_h} e^{-\left(\frac{1}{\lambda_h} - \beta_h\right)(t-b)} \left\{ \int_{t}^{+\infty} e^{-\int_t^z \left[\hat{r}(v) + \frac{1}{\lambda_h} - \beta_h\right] dv} \hat{\omega}^h(z) \tilde{N}^h(z)\, dz \right\} db$$

$H_\alpha(t)$ and $H_\beta(t)$ are rewritten as:

$$H_\alpha(t) = \int_{t}^{+\infty} e^{-\int_t^z \left[\hat{r}(v) + \frac{1}{\lambda_h} - \alpha_h\right] dv} \left\{ \int_{-\infty}^{t} a_h e^{\alpha_h(t-b)} \frac{1}{\lambda_l + \lambda_h} e^{-\left(\frac{1}{\lambda_h}\right)(t-b)} \hat{\omega}^h(z) \tilde{N}^h(z)\, db \right\} dz$$

$$H_\beta(t) = \int_{t}^{+\infty} e^{-\int_t^z \left[\hat{r}(v) + \frac{1}{\lambda_h} - \beta_h\right] dv} \left\{ \int_{-\infty}^{t} e^{\beta_h(t-b)} \frac{1}{\lambda_l + \lambda_h} e^{-\left(\frac{1}{\lambda_h}\right)(t-b)} \hat{\omega}^h(z) \tilde{N}^h(z)\, db \right\} dz$$

Therefore the two components of human wealth become:

$$H_\alpha(t) = \int_t^{+\infty} e^{-\int_t^z \left[\hat{r}(v)+\frac{1}{\lambda_h}-\alpha_h\right]dv} \frac{a_h}{\frac{1}{\lambda_h}-\alpha_h} \frac{1}{\lambda_l+\lambda_h} \hat{\omega}^h(z)\tilde{N}^h(z)\,dz$$

$$H_\beta(t) = \int_t^{+\infty} e^{-\int_t^z \left[\hat{r}(v)+\frac{1}{\lambda_h}-\beta_h\right]dv} \frac{1}{\frac{1}{\lambda_h}-\beta_h} \frac{1}{\lambda_l+\lambda_h} \hat{\omega}^h(z)\tilde{N}^h(z)\,dz$$

Differentiating with respect to time yields:

$$\dot{H}_\alpha(t) = \frac{a_h}{\frac{1}{\lambda_h}-\alpha_h} \frac{1}{\lambda_l+\lambda_h} \left\{ \left[\hat{r}(v)+\frac{1}{\lambda_h}-\alpha_h\right] H_\alpha(t) - \hat{\omega}^h(t)\tilde{N}^h(t) \right\}$$

$$\dot{H}_\beta(t) = \frac{1}{\frac{1}{\lambda_h}-\beta_h} \frac{1}{\lambda_l+\lambda_h} \left\{ \left[\hat{r}(v)+\frac{1}{\lambda_h}-\beta_h\right] H_\beta(t) - \hat{\omega}^h(t)\tilde{N}^h(t) \right\}$$

Implicitly, a_h, α_h, β_h verify:

$$\int_{-\infty}^t \frac{1}{\lambda_l+\lambda_h} e^{-\frac{1}{\lambda_h}(t-b)} \left[a_h \exp^{\alpha_h(t-b)} - \exp^{\beta_h(t-b)}\right] db = 1$$

a_h is therefore given by:

$$\left(\frac{1}{\lambda_l+\lambda_h}\right)\left(\frac{a_h}{\frac{1}{\lambda_h}-\alpha_h} - \frac{1}{\frac{1}{\lambda_h}-\beta_h}\right) = 1$$

Note that for low income earners, the distribution of efficient labor is given by:

$$\tilde{N}^l(b,t) = \left[a^l e^{\alpha^l(t-b)} - e^{\beta^l(t-b)}\right] E^l(t) N^l(t)$$

Implicitly, a_l, α_l, β_l verify:

$$\int_{-\infty}^t \frac{1}{\lambda_h} e^{-\left(\frac{1}{\lambda_l}+\frac{1}{\lambda_h}\right)(t-b)} \left[a_l \exp^{\alpha_l(t-b)} - \exp^{\beta_l(t-b)}\right] db = 1$$

a_l is therefore given by:

$$\frac{1}{\lambda_h} \frac{a_l}{\left(\frac{1}{\lambda_l}+\frac{1}{\lambda_h}\right)-\alpha_l} - \frac{1}{\lambda_h} \frac{1}{\left(\frac{1}{\lambda_l}+\frac{1}{\lambda_h}\right)-\beta_l} = 1$$

Appendix C

In the steady state, the concavity of F implies that:

$$F(K,N) > F_K^{'}(t)\,K$$

equivalently:

$$C^h + C^l + F_K^{'}(t)\,T_k K + F_{\tilde{N}}^{'}(t)\,T_\omega \tilde{N} > F_K^{'}(t)\,K$$

Replacing C^h and C^l by their expressions and reorganizing, this inequality becomes:

$$\left(\delta + \frac{1}{\lambda_h} - F_K^{'}(t)\,(1-T_k(t))\right) K^h + F_{\tilde{N}}^{'}(t)\,\tilde{N}(t) + \left(\frac{\delta - F_K^{'}(t)\,(1-T_k(t))}{F_K^{'}(t)\,(1-T_k(t)) + \frac{1}{\lambda_h}}\right)(1-T_\omega(t))\,F_{\tilde{N}}^{'}(t)\,\tilde{N}^h(t) > 0$$

We considering that $T_\omega(t) < 100\%$. Based on the dynamic equation for consumption, $F_K^{'}(t)\,(1-T_k(t)) > \delta$ in the steady state. Then, for this inequality to hold, a sufficient condition is::

$$F_K^{'}(t)\,1 - T_k(t) < \delta + \frac{1}{\lambda_h}$$

Appendix D

Deriving aggregate consumption, and using aggregate human and non-human wealth, I obtain an expression of the dynamics of aggregate consumption:

$$\dot{C}^h(t) = \Theta C^h(t) - \Xi W^h(t) + \Omega H^h_\beta(t) \tag{1}$$

with:

$$\Theta = \frac{1}{\lambda_h}\frac{a_h}{\frac{1}{\lambda_h} - \alpha_h}\left[\hat{r}(v) + \frac{1}{\lambda_h} - \alpha_h\right] - \varphi^h(t) \tag{2}$$

$$\Xi = \varphi^h(t)\left\{\frac{1}{\lambda_h}\frac{a_h}{\frac{1}{\lambda_h} - \alpha_h}\left[\hat{r}(v) + \frac{1}{\lambda_h} - \alpha_h\right] - \hat{r}(v)\right\} \tag{3}$$

$$\Omega = \frac{1}{\lambda_h}\frac{a_h}{\frac{1}{\lambda_h} - \alpha_h}\left[\hat{r}(v) + \frac{1}{\lambda_h} - \alpha_h\right] - \frac{1}{\lambda_h}\frac{1}{\frac{1}{\lambda_h} - \beta_h}\left[\hat{r}(v) + \frac{1}{\lambda_h} - \beta_h\right] \tag{4}$$