OPTIMAL TAXATION AND FINITE HORIZON - PROOFS -

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The Appendix is organized as follows:

Appendix 1: The primal approach (section 2)
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1 The primal approach

Definition 1 ***The Ramsey Equilibrium***

Given $\{G(z)\}_{z=t}^{+\infty}$ *a sequence of public spending. Given initial aggregate endowment* $W(t)$ *and individual endowment* $\{W(s,t)\}_{s=-\infty}^{t}$ *such that:*

$W(t) = B(t) + K(t)$ *et* $W(t) = \int_{-\infty}^{t} \lambda e^{(-\lambda(t-s))} W(s,t)ds,$

A competitive equilibrium (Ramsey equilibrium) is composed of allocations $\{\{C(s,z), L(s,z)\}_{s=-\infty}^{+\infty}\}_{z=t}^{+\infty}$ *and* $\{K(z)\}_{z=t}^{+\infty}$*, of a fiscal policy arrangement* $\pi = \{T_\omega(s,z), T_k(s,z)\}_{s=-\infty}^{+\infty}, B(z)\}_{z=t}^{+\infty}$*, of a price system* $p = \{w(z), r(z)\}_{z=t}^{+\infty}$*, and a sequence of asset holdings* $\{W(s,z)\}_{s=-\infty}^{+\infty}\}_{z=t}^{+\infty}$*, such that:*

1) Given prices and taxes, the allocation $\{C(s,z), L(s,z), W(s,z)\}_{s=-\infty}^{+\infty}$ *solves the consumer's problem for* $z \in [t, +\infty[$*;*

2) The allocation $\{L(z), K(z)\}_{z=t}^{+\infty}$ *solves the firm's problem. Factor prices are competitive:*

$r(z) + d = F'_{K(z)}$ *and* $\omega(z) = F'_{N(z)}$*, for* $z \in [t, +\infty[$*;*

3) The government's budget constraint is satisfied for $z \in [t, +\infty[$*;*

4) The allocation $\{\{C(z), K(z)\}_{z=t}^{+\infty}$ *satisfies the feasibility constraint:*

$\dot{K}(z) = F'_{K(z)}K(z) + F'_{N(z)}N(z) - dK(z) - C(z) - G(z)$ *for* $z \in [t, +\infty[$*;*

According to the primal approach, we re-formulate the problem of choosing optimal tax rates into a problem of choosing optimal allocations, called the Ramsey problem:

Proposition 1: *An allocation* $\{\{C(s,z), L(s,z)\}_{s=-\infty}^{+\infty}, K(z)\}_{z=t}^{+\infty}$*, is implementable for the government if it satisfies the implementability constraint:*

- for an agent born on date $s < t$:

$$\int_t^{+\infty} e^{-(\delta+\lambda)(z-s)} \left(U_{C(s,z)} C(s,z) + U_{N(s,z)} N(s,z)\right) dz = e^{-(\delta+\lambda)(t-s)} U_{C(s,t)} W(s,t)$$

- for an agent to be born on date $s > t$:

$$\int_s^{+\infty} e^{-(\delta+\lambda)(z-s)} \left(U_{C(s,z)} C(s,z) + U_{N(s,z)} N(s,z)\right) dz = U_{C(s,s)} W(s,s)$$

and the feasibility constraint :

$$\dot{K}(z) = Y(z) - dK(z) - C(z) - G(z)$$

for $z \in [t, +\infty[$

Proof of proposition 1:

We prove propositions 2 and 3, in order to prove proposition 1. First, we show that by construction, an allocation which satisfies the first-order conditions of the consumer's problem satisfies the implementability constraint (proposition 2).

Second, we show that an allocation that satisfies the implementability and the feasibility constraint can be decentralised as a competitive equilibrium in an age-dependent tax system (proposition 3). (We follow the same procedure as Chari and Kehoe [1998], and Gervais and Erosa [2002]).

Proposition 2: *By construction, an allocation* $\left\{\{C(s,z), L(s,z), W(s,z)\}_{s=-\infty}^{+\infty}\right\}_{z=t}^{+\infty}$ *which satisfies the first-order conditions of the consumer's problem satisfies the implementability constraint.*

Proof of proposition 2

We start writing the first-order conditions on date z for an individual born on date s. We assume that those are necessary and sufficient conditions and that the allocations are interior (assuming that the utility function is concave, the first-order conditions are sufficient and interiority is ensured by monotony and Inada conditions):

$$\mu(s,z) = U_{C(s,z)}$$
$$\mu(s,z)\hat{\omega}(s,z) = -U_{N(s,z)}$$
$$\dot{\mu}(s,z) = [\delta - \hat{r}(s,z)]\,\mu(s,z)$$

We build the consumer's intertemporal budget constraint:

$$\int_t^{+\infty} R_\lambda(z) \left[C(s,z) - \omega(z)\left(1 - T\omega(s,z)\right) N(s,z)\right] dz = W(s,t)$$

Verified for $z \in [t, +\infty[$.

Substituting the first-order conditions in the consumer's intertemporal budget constraint,we get the implementability constraint for an indiviual born on date $s < t$:

$$\int_t^{+\infty} e^{-(\delta+\lambda)(z-s)} \left[U_{C(s,z)} C(s,z) + U_{N(s,z)} N(s,z)\right] dz = e^{-(\delta+\lambda)(t-s)} U_{C(s,t)} W(s,t)$$

In the same way, we obtain the implementability constraint for an individual born on date $s > t$:

$$\int_{s}^{+\infty} e^{-(\delta+\lambda)(z-s)} \left[U_{C(s,z)}C\left(s,z\right) + U_{N(s,z)}N\left(s,z\right)\right] dz = U_{C(s,s)}W\left(s,s\right)$$

Proposition 3: *If an allocation* $\{\{C(s,z), L(s,z)\}_{s=-\infty}^{+\infty} K(z)\}_{z=t}^{+\infty}$ *satisfies the implementability and feasibility constraints, then a price system* p*, a fiscal policy* π*, asset holdings* $\{W(s,z)\}_{s=-\infty}^{+\infty}\}_{z=t}^{+\infty}$*, and an allocation* $\{K(z)\}_{z=t}^{+\infty}$ *constitute, with the given allocation, a competitive equilibrium in a fiscal system where taxes are age- dependent.*

Proof of proposition 3:

1) Suppose the allocation $\{\{C(s,z), L(s,z), W(s,z)\}_{s=-\infty}^{+\infty}\}_{z=t}^{+\infty}$ satisfies the implementability constraint:

$$\int_{t}^{+\infty} e^{-(\delta+\lambda)(z-s)} \left(U_{C(s,z)}C\left(s,z\right) + U_{N(s,z)}N\left(s,z\right)\right) dz = e^{-(\delta+\lambda)(t-s)}U_{C(s,t)}W\left(s,t\right)$$

Using the first-order conditions of the firm, $r(z) + d = F'_{K(z)}$ and $\omega(z) = F'_{N(z)}$, for $z \in [t, +\infty[$, we define a price system p.

We build the fiscal policy π using the sequence of after tax prices $\{\hat{\omega}(s,z), \hat{r}(s,z)\}_{s=-\infty}^{+\infty}$:

$$\delta - \hat{r}(s,z) = \frac{\dot{U}_{C(s,z)}}{U_{C(s,z)}} \quad (1)$$

$$\hat{\omega}(s,z) = \frac{-U_{N(s,z)}}{U_{C(s,z)}} \quad (2)$$

for $z \in [t, +\infty[$.

When $U_{C(s,z)} = \mu(s,z)$, for $z \in [t, +\infty[$ and $s =]-\infty, +\infty[$, by construction, $\{C(s,z), L(s,z)\}_{s=-\infty}^{+\infty}$ satisfies the consumer's first-order conditions on all dates $z \in [t, +\infty[$.

The proposition that an implementable allocation can be decentralized as a competitive equilibrium only if it satisfies the implementability constraint is valid only in an age-dependent tax system[1]. We observe that on date z, an allocation satisfies the implementability constraint if it satisfies (1) and (2). The optimal tax policy is therefore based on age-dependent after-tax prices and age-dependent Marginal Rates of Substitution (MRS). In other words, an allocation is coherent with the first-order conditions of consumers only in an age-dependent tax system.

Supposing that the first-order conditions are satisfied, we now show that the consumer budget constraint and the government budget constraint are also satisfied. Then we show that the transversality condition is also satisfied.

[1]This has been explicited by Erosa and Gervais [2001].

2) Given asset holdings $\{\{W(s,z)\}_{s=-\infty}^{+\infty}\}_{z=t}^{+\infty}$, we rewite the consumer's budget constraint taking into consideration the expressions of after-tax prices given by (1) and (2), then:

$$\dot{W}(s,z) = (\lambda + \hat{r}(s,z))\,W(s,z) + \hat{\omega}(s,z)N(s,z) - C(s,z)$$

Integrating and using the other first-order conditions of the consumer, we get:

$$\int_t^{+\infty} e^{-(\delta+\lambda)(z-s)}\left[U_{C(s,z)}C(s,z) + U_{N(s,z)}N(s,z)\right]dz = -[U_{C(s,z)}e^{-(\delta+\lambda)(z-s)}W(s,z)]_{z=t}^{+\infty}$$

The respect of the implementability constraint implies:

$$\lim_{z\to+\infty} \mu(s,z)\,W(s,z)\,e^{-(\delta+\lambda)(z-s)} = 0$$

The transversality condition is satisfied for all s. The optimality conditions of the consumer's problem are therefore all satisfied.

We now show that the respect of the implementability and feasibility constraints lead to respect the government budget constraint.

3) Assuming that the allocation $\{C(s,z), L(s,z), K(z)\}_{s=-\infty}^{+\infty}$ satisfies the implementability and feasibility constraint, then the aggregate consumer budget constrained is verified. As the production function is homogeneous of degree 1, by Walras law, we deduct that the governement budget constraint is satisfied.

2 The υ function

According to the primal approach, the implementability constraint is included in the individual's objective (IW). Consequently, the individual maximizes a "pseudo welfare" function that includes his utility (U) and the implementability constraint. $\chi(s)$ is the multiplier associated with the implementability constraint of individuals born in s.

Traditionally the individual welfare function is written:

$$IW(s,t) = \int_t^{+\infty} e^{-(\lambda+\delta)(z-s)}\upsilon_{traditional}\left(C(s,z), 1-N(s,z)\right)dz - \chi(s)e^{-(\delta+\lambda)(t-s)}U_{C(s,t)}W(s,t)$$

with:

$$\upsilon_{traditional}\left(C(s,z), 1-N(s,z)\right) = U\left(C(s,z), 1-N(s,z)\right) + \chi(s)\left(U_{C(s,z)}C(s,z) + U_{N(s,z)}N(s,z)\right)$$

The optimality conditions of the government problem are usually given for the dates after the initial date of the plan $(z > 0)$ and they do not include the term describing the initial conditions: $e^{-(\delta+\lambda)(t-s)}U_{C(s,t)}W(s,t)$, which is only present on date t. This term reflects that individuals start with a positive wealth.

We replace the traditional pseudo welfare funtion $\upsilon_{traditional}$ with υ:

$$\upsilon\left(C(s,z), 1-N(s,z)\right) = U\left(C(s,z), 1-N(s,z)\right) + \chi(s)\left[\begin{array}{c}\left(U_{C(s,z)}C(s,z) + U_{N(s,z)}N(s,z)\right) \\ -e^{-(\lambda+\delta)(z-t)}U_{C(s,t)}W(s,t)\end{array}\right]$$

3 The solution to the government's problem

On date z, the government choses the optimal allocation of aggregate consumption and labor between individuals of all ages in order to maximize their instantaneous pseudo utility υ. Given this first step, the government chooses an optimal allocation path (for consumption, labor and capital) maximizing its objective function under a feasibility constraint.

We solve the government's static and dynamic problems each time using a standard hamiltonian.

- The static problem: On date z, the government allocates aggregate consumption and labor among individuals of all ages.

$V(C(z), N(z)) = \max_{\{C(z-n,z),N(z-n,z)\}_{n=0}^{+\infty}} \int_0^{+\infty} \left[e^{-\lambda n}\upsilon(C(z-n,z), 1-N(z-n,z))\right] e^{(\rho-\delta)n} dn$

s.t.: $C(z) = \int_0^{+\infty} \lambda e^{-\lambda n} C(z-n,z)\, dn$ and $N(z) = \int_0^{+\infty} \lambda e^{-\lambda n} N(z-n,z)\, dn$

Given ψ_1 and ψ_2, the multipliers associated with the agregation constraints, the optimality conditions of the static problem are:

$$\upsilon_{C(z-n,z)} e^{(\rho-\delta)n} = -\psi_1(z)\lambda$$

$$\upsilon_{N(z-n,z)} e^{(\rho-\delta)n} = -\psi_2(z)\lambda$$

for $n \in [0; +\infty[$ and for $\delta \leq \rho < \delta + \lambda$.

The dynamic problem: Given the optimal allocations determined by the static problem, the government chooses an optimal allocation path for all z.

$$\max_{\{\{C(z),N(z)\},K(z)\}_{z=0}^{+\infty}} \int_0^{+\infty} \{V(C(z), 1-N(z))\} e^{-\rho z} dz$$

s.t.: $\dot{K}(z) = F'_{K(z)} K(z) + F'_{N(z)} N(z) - dK(z) - C(z) - G(z)$

Given ψ, the multiplier associated with the feasibility constraint, the necessary conditions of the dynamic problem are:

$$V_{C(z)} = \psi(z)$$

$$V_{N(z)} = -\psi(z)\omega(z)$$

$$\dot{\psi}(z) = \left[\rho - F'_K(z)\right]\psi(z)$$

The implementability constraint implies that the transversality condition is verified for each individual. Since the implementability constraint is part of the government's objective, the transversality condition is verified at the aggregate level.

Proposition 4: Both problems (static and dynamic) are re-united using the result by Benveniste and Sheinkman [1979]: $V(\cdot)$ is strictly concave and can be considered as the value function of a dynamic problem in which z represents time. $V(\cdot)$ is continuously differentiable in $C(z)$ and $N(z)$ and at the optimum: $V_{C(z)} = \upsilon_{C(z,z)}$ and $V_{N(z)} = \upsilon_{N(z,z)}$.

The necessary conditions for a static optimum are thus rewritten[2]:

$$\upsilon_{C(z-n,z)} e^{(\rho-\delta)n} = \psi(z) \text{ and } \upsilon_{N(z-n,z)} e^{(\rho-\delta)n} = -\psi(z)\,\omega(z)$$

The necessary condition for an optimal path$\{\{C(z), N(z)\}, K(z)\}_{z=0}^{+\infty}$ is:

$$\dot{\psi}(z) = \left[\rho - F_K^{'}(z)\right]\psi(z)$$

for $z \in [0; +\infty[$, $n \in [0; +\infty[$ and $\delta \leq \rho < \delta + \lambda$. The transversality condition is verified at the aggregate level.

4 The optimal tax rates

The capital income tax is derived from the difference between private and social MRS between current and future consumption:

$$\frac{\dot{U}_{C(z-n,z)}}{U_{C(z-n,z)}} - \frac{\dot{V}_{C(z-n,z)}}{V_{C(z-n,z)}} = \delta - \hat{r}(z-n,z) - (\rho - r(z))$$

Since:

$$\frac{\dot{V}_{C(z-n,z)}}{V_{C(z-n,z)}} = \frac{\dot{\upsilon}_{C(z-n,z)}}{\upsilon_{C(z-n,z)}} + \rho - \delta$$

Then:

$$\frac{\dot{U}_{C(z-n,z)}}{U_{C(z-n,z)}} - \frac{\dot{\upsilon}_{C(z-n,z)}}{\upsilon_{C(z-n,z)}} = r(z) - \hat{r}(z-n,z)$$

[2] We adopt the following notation :
$\upsilon_{C(z-n,z)} = [1 + \chi(z-n) + \chi(z-n)\xi^C(z-n,z)]U_{C(z-n,z)}$
$\quad - \frac{\partial}{\partial C(z-n,z)}\left[\chi(z-n)U_{C(-n,0)}W(-n,0)\right]$
$\upsilon_{N(z-n,z)} = [1 + \chi(z-n) + \chi(z-n)\xi^N(z-n,z)]U_{N(z-n,z)}$
$\quad - \frac{\partial}{\partial N(z-n,z)}\left[\chi(z-n)U_{C(-n,0)}W(-n,0)\right]$
ξ^C and ξ^N denote the "general equilibrium elasticities" (Atkeson, Chari and Kehoe (1999)):
$\xi^C(z-n,z) = \frac{U_{C(z-n,z)C(z-n,z)}C(z-n,z) + U_{N(z-n,z)C(z-n,z)}N(z-n,z)}{U_{C(z-n,z)}}$
$\xi^N(z-n,z) = \frac{U_{C(z-n,z)N(z-n,z)}C(z-n,z) + U_{N(z-n,z)N(z-n,z)}N(z-n,z)}{U_{N(z-n,z)}}$

We use the expression:
$v_{C(z-n,z)} = [1+\chi(z-n)+\chi(z-n)\xi^C\,(z-n,z)]U_{C(z-n,z)} - \frac{\partial}{\partial C(z-n,z)}\left[\chi(z-n)U_{C(-n,0)}W(-n,0)\right]$

On dates $z>0$:
$v_{C(z-n,z)} = [1+\chi(z-n)+\chi(z-n)\xi^C\,(z-n,z)]U_{C(z-n,z)}$

Following Chari and Kehoe (1998), we observe that the state of the economy follows a Markov process in continuous time with a unique and invariant probability density. Consequently, the optimal allocations on dates $z>0$ are time invariant. On date $z=0$, however, the allocation rules include terms relative to the financial wealth of agents born before date zero:

On date $z=0$:
$v_{C(-n,0)} = [1+\chi(-n)+\chi(-n)\xi^C\,(-n,0)]U_{C(-n,0)} - \frac{\partial}{\partial C(-n,0)}\left[\chi(-n)U_{C(-n,0)}W(-n,0)\right]$
Equivalently:
$v_{C(-n,0)} = [1+\chi(-n)+\chi(-n)\xi^C\,(-n,0)]U_{C(-n,0)} - \left[\chi(-n)U_{C(-n,0)C(-n,0)}W(-n,0)\right]$

Also, on dates $z>0$:
$\dot{v}_{C(z-n,z)} = \dot{U}_{C(z-n,z)}\,[1+\chi(z-n)+\chi(z-n)\xi^C\,(z-n,z)]$
$\quad +U_{C(z-n,z)}\chi(z-n)\dot{\xi}^c\,(z-n,z)$

On date zero:
$\dot{v}_{C(-n,0)} = \dot{U}_{C(-n,0)}\,[1+\chi(-n)+\chi(-n)\xi^C\,(-n,0)]$
$\quad +U_{C(-n,0)}\chi(-n)\dot{\xi}^c\,(-n,0)$
$\quad -\chi(-n)\,\dot{U}_{C(-n,0)C(-n,0)}\,W(-n,0) - \chi(-n)U_{C(-n,0)C(-n,0)}\,\dot{W}\,(-n,0)$

We calculate the optimal capital income tax on date zero and derive the general expression for all dates $z \geq 0$. After simplifications, we obtain:

$$r(z) - \hat{r}(z-n,z) = -\frac{\dot{\xi}^c\,(z-n,z)\,\chi(z-n)}{[1+\chi(z-n)+\chi(z-n)\xi^c\,(z-n,z)]} + Ini3$$

with: $Ini3 = 0$ for $z>0$, and:
$Ini3 = (r\,(0)-\delta)\frac{\chi(-n)W(-n,0)U_{C(-n,0)C(-n,0)}}{U_{C(-n,0)}[1+\chi(-n)+\chi(-n)\xi^c(-n,0)]} + \frac{\chi(-n)\left(\dot{U}_{C(-n,0)C(-n,0)}W(-n,0)+U_{C(-n,0)C(-n,0)}\dot{W}(-n,0)\right)}{U_{C(-n,0)}[1+\chi(-n)+\chi(-n)\xi^c(-n,0)]}$
for $z=0$.

Following Erosa and Gervais 2001, we consider that derivative of χ with respect to time is equal to zero: $\dot{\chi}\,(z-n)=0$. In a representative agent model, this multiplier representing the "deadweight loss" of the distortionary tax system is unique and constant over time. In a generations model, there is an implementability constraint for each date $s=z-n$. There is therefore one multiplier per date s. Also, there is one single implementability constraint on a given date for all individuals of the same age.

The labor income tax is derived from the ratio between private and social MRS between consumption and leisure:

$$\frac{-U_{N(z-n,z)}/U_{C(z-n,z)}}{-V_{N(z-n,z)}/V_{C(z-n,z)}} = \frac{\hat{\omega}(z-n,z)}{\omega(z-n,z)} = (1 - T_{\omega(z-n,z)})$$

We use the expression:
$v_{N(z-n,z)} = [1+\chi(z-n)+\chi(z-n)\xi^N(z-n,z)]U_{N(z-n,z)} - \frac{\partial}{\partial N(z-n,z)}\left[\chi(z-n)U_{C(-n,0)}W(-n,0)\right]$

On date $z = 0$:
$v_{N(-n,0)} = [1+\chi(-n)+\chi(-n)\xi^N(-n,0)]U_{N(-n,0)} - \frac{\partial}{\partial N(-n,0)}\left[\chi(-n)U_{C(-n,0)}W(-n,0)\right]$
Equivalently:
$v_{N(-n,0)} = [1+\chi(-n)+\chi(-n)\xi^N(-n,0)]U_{N(-n,0)} - \chi(-n)U_{C(-n,0)N(-n,0)}W(-n,0)$

On dates $z > 0$:
$v_{N(z-n,z)} = [1+\chi(z-n)+\chi(z-n)\xi^N(z-n,z)]U_{N(z-n,z)}$

We calculate the optimal labor income tax on date zero and derive the general expression for all dates $z \geq 0$. After simplifications, we obtain:

$$T_\omega(z-n,z) = \frac{\chi(z-n)\left(\xi^N(z-n,z) - \xi^C(z-n,z)\right) + Ini1}{1+\chi(z-n)+\chi(z-n)\xi^N(z-n,z) + Ini2}$$

with: $Ini1 = 0$ and $Ini2 = 0$ for $z > 0$, and:
$Ini1 = \chi(-n)W(-n,0)\left(\frac{U_{C(-n,0)C(-n,0)}}{U_{C(-n,0)}} - \frac{U_{C(-n,0)N(-n,0)}}{U_{N(-n,0)}}\right)$
$Ini2 = -\chi(-n)W(-n,0)\frac{U_{C(-n,0)N(-n,0)}}{U_{N(-n,0)}}$
for $z = 0$.