251 Abstract Algebra - Midterm 2 - Solutions

Question 1

Let $\phi : G \to H$ be a homomorphism and let E be a subgroup of H.

(a) Prove that $\phi^{-1}(E) \leq G$.

(b) Let $N = \langle r^2 \rangle$ be a subgroup of D_8. You may assume $N \trianglelefteq D_8$. Consider the projection homomorphism $\phi : D_8 \to D_8/N$, which maps g to gN for each $g \in D_8$. What is $\phi^{-1}(\langle sN \rangle)$?

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Solution.

(a) Since E is a subgroup of H, we have $1_H \in E$. Therefore $1_G \in \phi^{-1}(E)$, so E is nonempty. Suppose that $x, y \in \phi^{-1}(E)$, then

$$\phi(xy^{-1}) = \phi(x)\phi(y^{-1}) = \phi(x)\phi(y)^{-1} \in E,$$

since $\phi(x) \in E$ and $\phi(y) \in E$, and since E is a subgroup we must have $\phi(y)^{-1} \in E$. Therefore, $xy^{-1} \in \phi^{-1}(E)$ and we see that $\phi^{-1}(E) \leq G$.

(b) The subgroup N has elements $\{1, r^2\}$, since $|r^2| = 2$. The subgroup $\langle sN \rangle$ has elements $1N = \{1, r^2\}$ and $sN = s, sr^2$, and nothing else since $|sN| = 2$. Therefore, the set $\phi^{-1}(\langle sN \rangle) = \{1, r^2, s, sr^2\}$. This is the subgroup $\langle s, r^2 \rangle$ of D_8.

Question 2

List all subgroups of S_3 and draw the subgroup lattice for S_3. Choose one non-trivial subgroup and give its normalizer.

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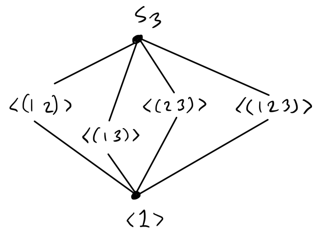
Solution. We have the subgroups:

$$\begin{aligned} &\langle 1 \rangle, \\ &\langle (1\ 2) \rangle, \\ &\langle (1\ 3) \rangle, \\ &\langle (2\ 3) \rangle, \\ \langle (1\ 2\ 3) \rangle = &\langle (1\ 3\ 2) \rangle, \end{aligned}$$

since cycles of length 2 have order 2, and since $(1\ 3\ 2) = (1\ 2\ 3)^2$. Let's find the normalizer of $\langle (1\ 2) \rangle$, which has elements $\{1, (1\ 2)\}$. Since $g1g^{-1} = 1$ always, we only need to check $g(1\ 2)g^{-1} \in \langle (1\ 2) \rangle$ to decide whether $g \in N_G(\langle (1\ 2) \rangle)$:

$$\begin{aligned} 1(1\ 2)1^{-1} &= (1\ 2), \\ (1\ 2)(1\ 2)(1\ 2)^{-1} &= (1\ 2), \\ (1\ 3)(1\ 2)(1\ 3)^{-1} &= (2\ 3), \\ (2\ 3)(1\ 2)(2\ 3)^{-1} &= (1\ 3), \\ (1\ 2\ 3)(1\ 2)(1\ 2\ 3)^{-1} &= (2\ 3), \\ (1\ 3\ 2)(1\ 2)(1\ 3\ 2)^{-1} &= (1\ 3). \end{aligned}$$

Therefore, we see that $N_G(\langle (1\ 2) \rangle) = \langle (1\ 2) \rangle$. The lattice diagram of S_3 is given by:



Question 3

Prove that if H is a subgroup of G, then $\langle H \rangle = H$.

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Solution. We have that

$$\langle H \rangle = \bigcap_{H \subseteq N \leq G} N,$$

which implies that $\langle H \rangle \subseteq H$, since $\langle H \rangle$ is the intersection of all subgroups N that contain H, and H is clearly one of them.

Conversely, since all of those subgroups N contain H by definition, their intersection must contain H and therefore $H \subseteq \langle H \rangle$.

Question 4

Lagrange's Theorem states that if G is a finite group and $H \leq G$, then the order of H divides the order of G. You may assume that the left cosets of H form a partition of G. From here, finish the proof of Lagrange's Theorem (i.e. show that all left cosets have the same number of elements).

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Solution. To show that all left cosets of H have the same number of elements, we show that they all have the same number of elements as H. Consider a left coset gH and the map $f : H \to gH$ given by $f(h) = gh$. All elements in gH are of the form gh for some $h \in H$, so this map is clearly invertible: $f^{-1}(gh) = g^{-1}(gh)$, which gives $|H| = |gH|$. Since the cosets form a partition of G with $|H|$ elements each, we see that $|H|$ must divide $|G|$.

Question 5

(a) Show that if G is an abelian group, every subgroup N of G is normal.

(b) Show that for any G (not necessarily abelian) if $N \leq Z(G)$ then N is normal.

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Solution.

(a) If G is abelian, then $gng^{-1} = gg^{-1}n = n$ for all $g \in G$ and $n \in N$. Therefore, $gNg^{-1} = gg^{-1}N = N$ for all $g \in G$, which implies that N is normal.

(b) In this case, we once again have that $gng^{-1} = gg^{-1}n = n$ for all $g \in G$ and $n \in N$, since $n \in Z(G)$ and commutes with every element of G (specifically g^{-1} in this case). Then the same conclusion holds.