251 Abstract Algebra - Midterm 2 Practice - Solutions

Question 1

Let H be a subgroup of G and fix some element $g \in G$.

(a) Prove that gHg^{-1} is a subgroup of G. [4 points]

(b) Prove that $|gHg^{-1}| = |H|$. [3 points]

(c) Describe the subgroup $s\langle r\rangle s^{-1}$ of D_8. [3 points]

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Solution.

(a) Suppose $x, y \in gHg^{-1}$. Then $x = gag^{-1}$ and $y = gbg^{-1}$ for some $a, b \in H$. Then

$$xy^{-1} = gag^{-1}(g^{-1})^{-1}b^{-1}g^{-1} = gag^{-1}(g^{-1})^{-1}b^{-1}g^{-1} = gab^{-1}g^{-1} \in gHg^{-1},$$

since $ab^{-1} \in H$. Since the subgroup criterion holds, gHg^{-1} is a subgroup of G.

(b) Consider the map $h \mapsto ghg^{-1}$ for $h \in H$, by left/right cancelation, we see that $h_1 = h_2 \Leftrightarrow gh_1g^{-1} = gh_2g^{-1}$, so the map is injective. It is also surjective since gHg^{-1} is exacty the set of elements that can be written as ghg^{-1} for $h \in H$. Since this describes a bijection between H and gHg^{-1}, we have $|gHg^{-1}| = |H|$.

(c) This is the subgroup on the set of elements $\{s1s^{-1}, srs^{-1}, sr^2s^{-1}, sr^3s^{-1}\} = \{1, r^3, r^2, r\}$. We see that $s\langle r\rangle s^{-1} = \langle r\rangle$. (This is true in general since $\langle r\rangle$ is a normal subgroup of D_8.)

Question 2

Prove that if H and K are both normal subgroups of G then their intersection $H \cap K$ is also a normal subgroup. [10 points]

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Solution. Define the homomorphism $\phi : G \to G/H \times G/K$ as $g \mapsto (gH, gK)$. Since G/H and G/K are both quotient groups this defines a homomorphism with

$$\phi(g_1g_2) = (g_1g_2H, g_1g_2K) = (g_1H, g_1K)(g_2H, g_2K) = \phi(g_1)\phi(g_2).$$

The kernel of ϕ is exactly those elements $g \in G$ such that $(gH, gK) = (1H, 1K) = (H, K)$, which is exactly when $g \in H \cap K$. Therefore, $H \cap K$ is a normal subgroup.

Question 3

Let H and K be subgroups of G. Draw all possible lattices on the set $G, 1, H, K, H \cap K, \langle H, K \rangle$. [10 points]

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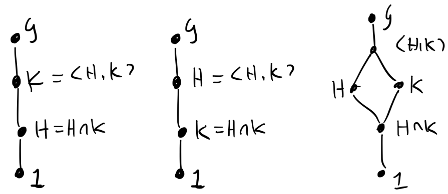
Solution. We distinguish between the cases $H \leq K$, $K \leq H$ and neither of those two, to get the following 3 scenarios:



Question 4

Consider the subgroup H of S_5 generated by (1 2) and (1 5).

(a) What is the order of H? [5 points]

(b) Is H normal in S_5? [5 points]

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Solution.

(a) We see that H has elements $(1\ 2), (1\ 5)$ and $(1\ 2)(1\ 5) = (1\ 5\ 2)$. We have seen that the elements $(a, b), (a, b, c)$ generate the full symmetric group on the set $\{a, b, c\}$. So, H has all permutations of $\{1, 2, 5\}$ and must therefore have 6 elements. (It cannot have more since no elements other than 1,2,5 are involved in (1 2) and (1 5).)

(b) H is not normal. For example, the element $(1\ 3)(1\ 2)(1\ 3)^{-1} = (2\ 3)$ is not in H.

Question 5

Let G be a group and suppose that $gNg^{-1} \subseteq N$ for all $g \in G$.

(a) Find a homomorphism $\phi : G \to G/N$ such that N is the kernel of ϕ. [6 points]

(b) Show that the left and right cosets of N induce the same partition of G. [4 points]

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Solution. First, we showed in question 1 that we have a bijection between N and gNg^{-1}, so we may conclude that $gNg^{-1} = N$.

(a) We let $\phi(g) = gN$. Then the kernel of ϕ is all elements of G such that $gN = 1N = N$, i.e. such that $g \in N$. We check that ϕ is well-defined:

$$\phi(g_1 g_2) = (g_1 g_2)N = g_1 g_2 NN = g_1 g_2 g_2^{-1} N g_2 N = g_1 N g_2 N = \phi(g_1)\phi(g_2).$$

(b) We saw that $gNg^{-1} = N$, which directly implies that $gN = Ng$, i.e. the left and right cosets are equal.