Question 1

Let $\sigma \in S_8$ be the following permutation:

$$\begin{array}{ll} 1 \mapsto 3 & 5 \mapsto 2 \\ 2 \mapsto 4 & 6 \mapsto 6 \\ 3 \mapsto 1 & 7 \mapsto 7 \\ 4 \mapsto 5 & 8 \mapsto 8. \end{array}$$

(a) Find the cycle decomposition of σ and σ^{-1} and write it in the standard format. [4 points]

(b) Find $|\sigma|$. [3 points]

(c) Find $(6\ 8\ 7)\sigma(6\ 7\ 8)$. [3 points]

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Solution.

(a) From the definition of σ above, we see that:

$$\sigma = (1\ 3)(2\ 4\ 5), \quad \sigma^{-1} = (1\ 3)(2\ 5\ 4).$$

(b) We see that σ has a cycle of length 2 and a cycle of length 3, and therefore $|\sigma| = lcm(2, 3) = 6$.

(c) This is the permutation composition

$$(6\ 8\ 7)\sigma(6\ 7\ 8) = (6\ 8\ 7)(1\ 3)(2\ 4\ 5)(6\ 7\ 8) = (1\ 3)(2\ 4\ 5) = \sigma.$$

Question 2

Prove that for a group G with $|G| = n > 2$ it is not possible to have a subgroup H with $|H| = n - 1$. [10 points]

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Solution. Suppose that G is a group with $|G| = n > 2$ and H is a subgroup with $|H| = n - 1$. Let $G \setminus H = \{x\}$. Since H is closed under inverses, we must have $x = x^{-1}$, since we cannot have $x^{-1} \in H$. Since $|G| > 2$ there must be another nonidentity element y not equal to x and therefore $y \in H$. Then the element xy is not equal to x and not equal to 1 (since $y \neq x^{-1}$), and therefore $xy \in H$. However by the Subgroup Criterion this implies that $(xy)y^{-1} \in H$, which implies $x \in H$. This is a contradiction, and we conclude that $|H| \neq n - 1$.

Question 3

For a group G and subset $A \subseteq G$, let $N_G(A)$ be the normalizer of A in G and $C_G(A)$ the centralizer of A in G. Show that $C_G(A) \leq N_G(A)$ and $Z(G) \leq N_G(A)$. [10 points]

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Solution. Suppose that $g \in C_G(A)$. Then $gag^{-1} = a$ for all $a \in A$. This implies that

$$gAg^{-1} = \{gag^{-1} \mid a \in A\} = \{a \mid a \in A\} = A,$$

and therefore $g \in N_G(A)$. This implies that $C_G(A) \leq N_G(A)$.

Suppose that $g \in Z(G)$. Then $ghg^{-1} = h$ for all $h \in G$. Since $A \subseteq G$ this implies that $gag^{-1} = a$ for all $a \in A$. Therefore, $g \in C_G(A) \leq N_G(A)$ (as we showed previously). This implies that $Z(G) \leq N_G(A)$.

Question 4

Consider the dihedral group D_{2n}:

$$D_{2n} = \langle r, s \mid r^n = s^2 = 1,\ rs = sr^{-1} \rangle.$$

Show that we can also write [10 points]

$$D_{2n} = \langle a, b \mid a^2 = b^2 = (ab)^n = 1 \rangle,$$

by letting $a = s$ and $b = sr$.

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Solution. Let $a = s$ and $b = sr$. Suppose that $r^n = s^2 = 1,\ rs = sr^{-1}$ hold. Then

$$\begin{aligned} a^2 &= s^1 = 1 \\ b^2 &= (sr)^2 = srsr = srr^{-1}s = s^2 = 1 \\ (ab)^n &= (ssr)^n = r^n = 1. \end{aligned}$$

Now, suppose that $a^2 = b^2 = (ab)^n = 1$ holds. We can rewrite: $s = a$ and $r = s^2r = ab$. Then

$$\begin{aligned} r^n &= (ab)^n = 1 \\ s^2 &= a^2 = 1 \\ rs = aba = ab^{-1}a^{-1} &= a(ab)^{-1} = sr^{-1}. \end{aligned}$$

Therefore, the two representations are equivalent.

Question 5

(a) For a group G acting on a set S. Let G_s be the stabilizer of $s \in S$ of the action. Show that $g \in G_s$ implies that $g^{-1} \in G_s$. (This is part of the proof of showing that the stabilizer is a subgroup of G.) [5 points]

(b) Let H be a subgroup of order 2 in G. Show that $N_G(H) = C_G(H)$. [5 points]

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Solution.

(a) Suppose $g \in G_s$. Then

$$g^{-1} \cdot s = g^{-1} \cdot (g \cdot s) = (g^{-1}g) \cdot s = 1 \cdot s \in G_s.$$

(b) Suppose H has the two distinct elements $1, a$. For any element $g \in G$, we have $g1g^{-1} = 1$, since 1 commutes with all elements. If $g \in N_G(H)$, then $\{g1g^{-1}, gag^{-1}\} = \{1, a\}$, and since $g1g^{-1} = 1$, we must have $gag^{-1} = a$ and therefore $g \in C_G(H)$. This implies that $N_G(H) \leq C_G(H)$. We showed in Q3 that $C_G(A) \leq N_G(A)$ and therefore $N_G(H) = C_G(H)$.